> These notes will grow over the course of the semester. They essentially mix lecture notes and links to other resources that may be helpful to look at. All of Robert Vandermolen's fancy slides with pictures that I didn't care to typeset myself can be found at `http://people.math.sc.edu/robertv/teaching.html`.
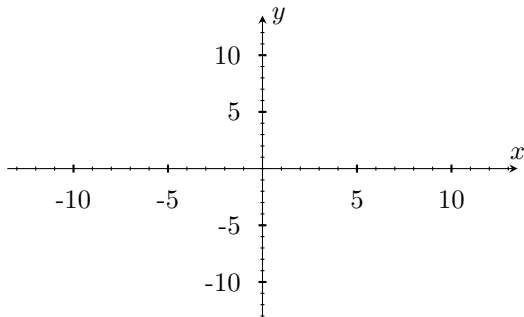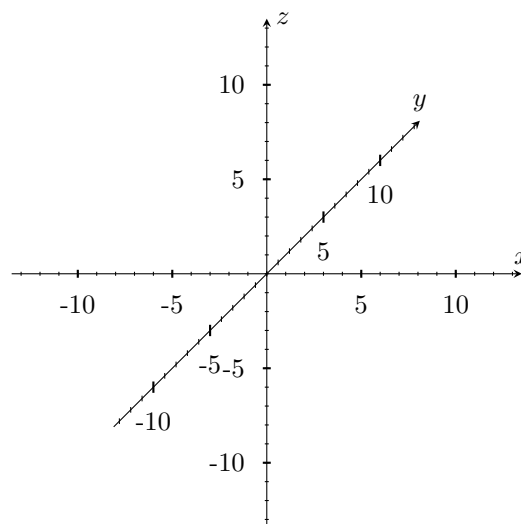
Contents

1   §12.1 Three-Dimensional Coordinate Systems

Whereas the plane only has two dimensions, with an x- and a y-axis, space has three dimensions, so we add a z-axis.

Plane

Space

To orient ourselves the same way in space, we always use the right-hand rule: If you curve the fingers of your right hand from the positive x-axis to the positive y-axis, then your thumb will point in the direction of the positive z-axis.

Since we have three variables, points are represented as triplets.

Plane
(x, y)

Space
(x, y, z)

We will discuss multivariable functions in later sections. Whereas we typically think of functions in the plane as being equations of the form $y = \langle$ expression $\rangle$, in space we usually solve for $z =$.

Plane
$y = 3x - \sin(x/2)$

Space
$z = \sin(x + y)$

But any equation in three variables satisfies a particular set of points in space, even if it isn't a function of z.

Plane
$x^2 + y^2 = 4$

Space
$x^2 + y^2 + z^2 = 9$

We can use online tools to visualize these objects.
`https://www.geogebra.org/3d?lang=en`

Distance and Spheres

In two dimensions, distance is calculated using Pythagorean's theorem.
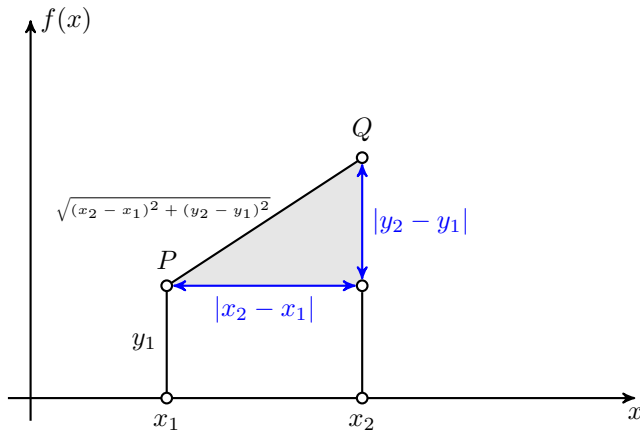
A similar formula holds in 3D (and in higher dimensions). Here's an example.

Example. *Find the distance between* $(2, 0, 2)$ *and* $(0, 3, 4)$.

Robert Vandermolen's 12.1 slides

DISTANCE!

How do we find the distance between two points?

$$|PQ|=?$$
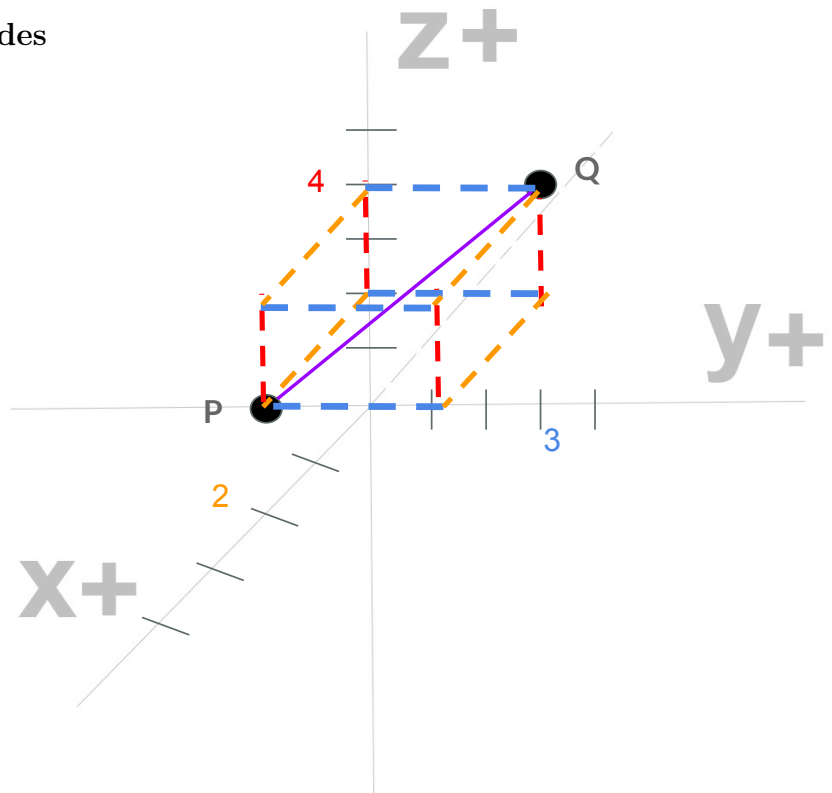
Let's try an example:

P=(2,0,2)

Q=(0,3,4)

DISTANCE!

How do we find the distance between two points?
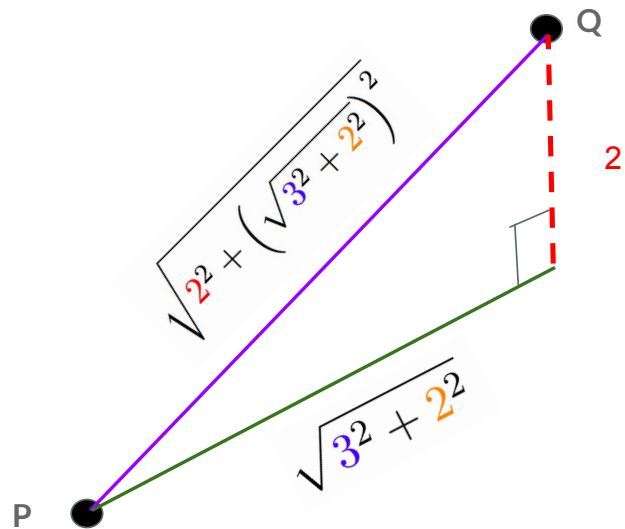
$$|PQ|=?$$

Let's try an example:

P=(2,0,2)

Q=(0,3,4)

Robert Vandermolen's 12.1 slides

DISTANCE!

How do we find the distance between two points?

    |PQ|=?

  Let's try an example:

    P=(2,0,2)

    Q=(0,3,4)

$$\sqrt{3^2 + 2^2}$$

2

2

P

3

Q

DISTANCE!

How do we find the distance between two points?

    |PQ|=?

  Let's try an example:

    P=(2,0,2)

    Q=(0,3,4)

$$\sqrt{2^2 + \left(\sqrt{3^2 + 2^2}\right)^2}$$

2

$$\sqrt{3^2 + 2^2}$$

P

Q

Robert Vandermolen's 12.1 slides

DISTANCE!

How do we find the distance between two points?

$|PQ|=?$

In General:

$P=(x,y,z)$

$Q=(x',y',z')$

$|z-z'|$

$|x-x'|$

$|y-y'|$

DISTANCE!

How do we find the distance between two points?

$|PQ|=?$

In General:

$P=(x,y,z)$

$Q=(x',y',z')$

$|z-z'|$

$\sqrt{(y-y')^2 + (x-x')^2}$

$|x-x'|$

$|y-y'|$

Robert Vandermolen's 12.1 slides

DISTANCE!

How do we find the distance between two points?

|PQ|=?

In General:

P=(x,y,z)

Q=(x',y',z')

$$\sqrt{(z-z')^2 + (y-y')^2 + (x-x')^2}$$

|z-z'|

Q

P

$$\sqrt{(y-y')^2 + (x-x')^2}$$
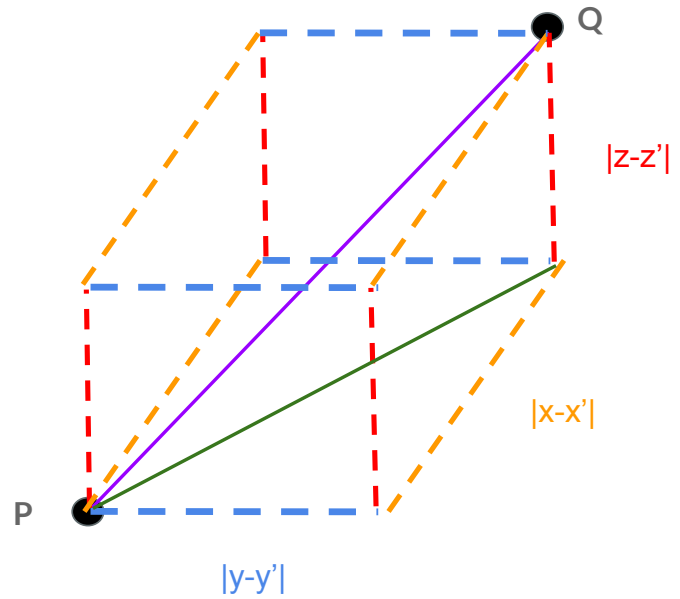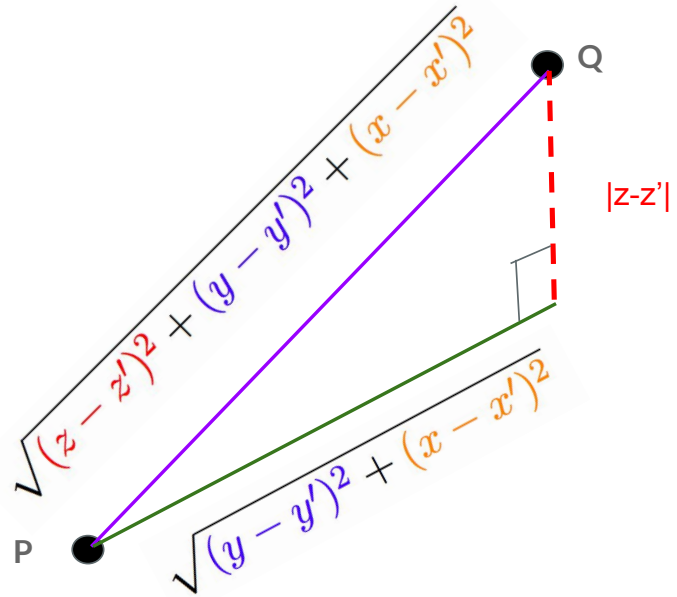
Thus we have the following distance formula.
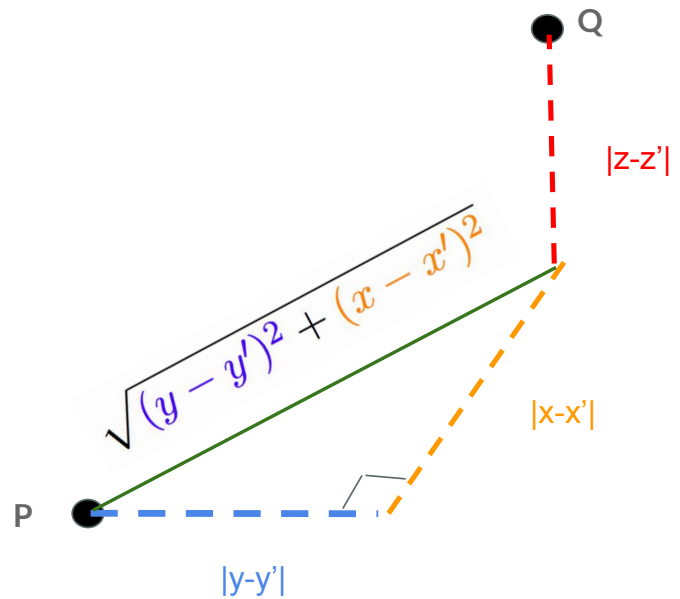
Definition. *The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is*

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

In the plane, a circle is all the points a fixed distance away from a fixed point.
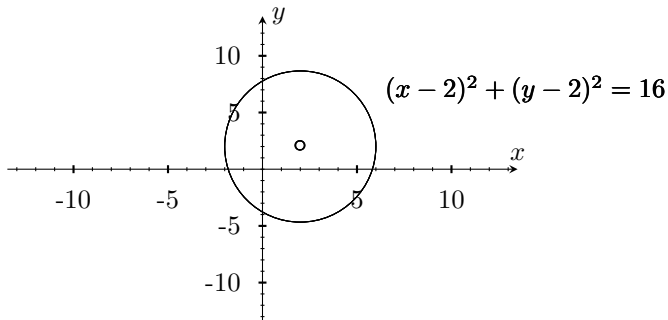
In space, the set of all points a fixed distance from a fixed point is a sphere.
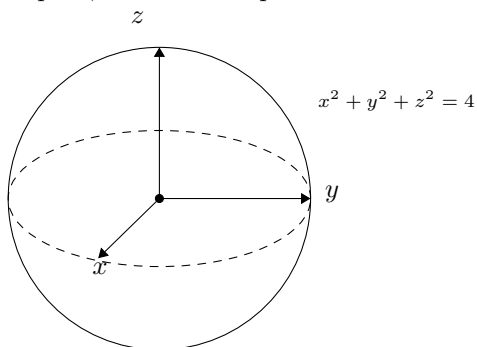
Using the distance formula where we make (x, y, z) a point on the sphere with center (x_0, y_0, z_0) and radius r, we get the standard equation for a sphere.

Definition. *The standard equation of a sphere with center (x_0, y_0, z_0) and radius r is*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Example. *Find the center and radius of $x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$.*

We complete the square in each variable, adding the same amount to both sides of the equation:

$$\left[x^2 + 3x + \left(\frac{3}{2}\right)^2\right] + y^2 + \left[z^2 - 4z + \left(-\frac{4}{2}\right)^2\right] = -1 + \left(\frac{3}{2}\right)^2 + \left(-\frac{4}{2}\right)^2$$

This simplifies to

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

Thus the sphere has center $\left(-\frac{3}{2}, 0, 2\right)$ and radius $\sqrt{21/4}$.

Links

Robert Vandermolen's 12.1 slides:
    `https://docs.google.com/presentation/d/1dJPXG1RYjUCIyv6xCgCFqbrHItoAOFa_z_2Z-yTje7w/edit#slide=id.gc6f83aa91_0_79`
    Graphing in 3D:
    `https://www.geogebra.org/3d?lang=en`

2 §12.2 Vectors

A vector is a directed line segment. You can think about a vector as an arrow pointing from one point to another point.

Put another way, a vector is the following information: A <u>direction</u> and a <u>length/magnitude</u>.
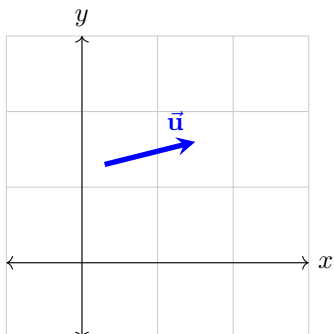

The direction is where the arrow points and the length is how long the arrow is.

A vector models any application where force is involved: velocity, displacement, work, etc.

However, for the purposes of equality, we don't care where a vector starts or ends. Thus the two vectors below are equal, since they have the same direction and length.
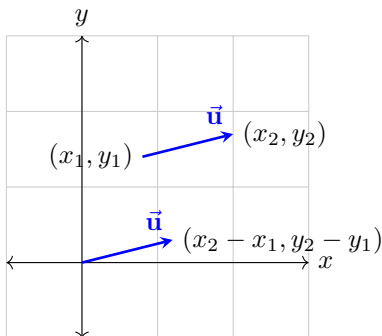
Since it doesn't matter where we draw a vector, we will usually place the initial point at the origin. This is called *standard position*.

Definition. *If a vector $\vec{v}$ goes from (x_1, y_1, z_1) to (x_2, y_2, z_2), then the same vector in standard position goes from $(0,0,0)$ to $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. In this case we write*

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Thus we identify every point in space with a vector from the origin to that point, and two vectors are equal if they have the same terminal point in standard position, i.e.,

$$\langle u_1, u_2, u_3 \rangle = \langle v_1, v_2, v_3 \rangle \iff u_1 = v_1, u_2 = v_2, \text{ and } u_3 = v_3$$

.

Example. *The vector $\vec{u} = \langle 2, -1, 4 \rangle$ is the vector that starts at the origin and goes to the point $(2, -1, 4)$.*

Example. *How would we write down the vector starting at $(-7, 5, 0)$ and going to $(3, -1, 4)$?*

$\vec{v} = \langle 10, -6, 4 \rangle.$

The length of a vector is simply the distance from its initial point to its terminal point.

Definition. *If $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then the length of $\vec{v}$ is*

$$\|\vec{v}\| = |\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Example. *The length of $\langle 7, 3, -2 \rangle$ is $\sqrt{49 + 9 + 4} = \sqrt{62}$.*

The only vector with length 0 is the vector that starts at a point and ends at the same point. We have special notation for this vector.

Definition. *The vector of all zeros is denoted $\vec{0} := \langle 0, 0, 0 \rangle$. This notation can be used no matter what dimension, i.e., in 2D we have $\vec{0} = \langle 0, 0 \rangle$.*

Vector Algebra

Given two vectors, we will now describe different ways to combine them, which will turn the set of vectors into something we can manipulate algebraically.

Definition. *If $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, and $k \in \mathbb{R}$, then we have the following operations:*
    Vector addition:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

    Scalar multiplication:

$$k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle.$$

This means we can add two vectors and multiply vectors by numbers. Geometrically, adding vectors corresponds to following first one vector and then the other to the resulting location. Scalar multiplication corresponds to stretching/shrinking a vector without changing its direction.
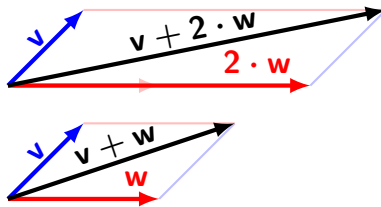
Example. *Let $\vec{v} = \langle 2, 4 \rangle$ and $\vec{u} = \langle -4, 6 \rangle$. Find the component form of $\frac{1}{2}(\vec{v} + \vec{u})$.*

$$\frac{1}{2}(\vec{v} + \vec{u}) = \frac{1}{2}(\langle 2, 4 \rangle + \langle -4, 6 \rangle) = \frac{1}{2}\langle -2, 10 \rangle = \langle -1, 5 \rangle.$$

Using just vector addition, scalar multiplication, and the properties of the real numbers, we get the following properties (page 712 of the textbook):

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

3. $\vec{u} + \vec{0} = \vec{u}$

4. $\vec{u} + (-\vec{u}) = \vec{0}$

5. $0\vec{u} = \vec{0}$

6. $1\vec{u} = \vec{u}$

7. $a(b\vec{u}) = (ab)\vec{u}$

8. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

9. $(a + b)\vec{u} = a\vec{u} + b\vec{u}$

Here's a proof of 3. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$.

$\vec{u} + \vec{0}$	Left hand side of equation
$= \langle u_1, u_2, u_3 \rangle + \langle 0, 0, 0 \rangle$	Definition of vectors
$= \langle u_1 + 0, u_2 + 0, u_3 + 0 \rangle$	Vector addition
$= \langle u_1, u_2, u_3 \rangle$	0 additive identity
$= \vec{u}$	Right hand side of equation

Cont.

Unit Vectors

We can change the length of a vector with scalar multiplication, so we are interested in vectors that have length 1. These vectors have a specific name.

Definition. *A vector $\vec{v}$ is a <u>unit vector</u> if its length is 1.*

Example. *$\vec{v} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ is a unit vector because $\|\vec{v}\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$.*

In 3D, we have special unit vectors which point in the direction of the $x-, y-$ and z-axes.

Definition. *The standard unit vectors are $\vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle$, and $\vec{k} = \langle 0, 0, 1 \rangle$.*

We can break up any vector as $\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$.

Note that we can always change a (nonzero) vector to a unit vector in the same direction by dividing by its length, that is, $\left(\frac{1}{\|\vec{v}\|} \right) \vec{v}$ is a unit vector in the direction of $\vec{v}$.

Links

Robert Vandermolen's 12.2 slides:
    `https://docs.google.com/presentation/d/1Vw6WT3sHAfts0ZrzP-5yjX37eaXP5Gk49jqckDkdXPI/edit#slide=id.gc6f83aa91_0_79`

3　§12.3 The Dot Product

Dot products

The *dot product* of two vectors gives us geometric information about the angle between the vectors.

Definition. *Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then*

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

is the dot product of $\vec{u}$ and $\vec{v}$.

The dot product of two vectors is a number, not a vector.

Example. $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3) = -7.$

Angles

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors. If θ is the angle between $\vec{u}$ and $\vec{v}$, then

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \right)$$

Two other ways of saying this are

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \qquad \text{or} \qquad \vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos(\theta).$$

Notice that the last formula means we can calculate a dot product of two vectors knowing <u>only</u> the lengths of the vectors and the angle between them.

Example. *Find the angle between $\vec{u} = \vec{i} - 2\vec{j} - 2\vec{k}$ and $\vec{v} = 6\vec{i} + 3\vec{j} + 2\vec{k}$.*

We have $\vec{u} \cdot \vec{v} = 1(6) - 2(3) - 2(2) = -4$, $\|\vec{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = 3$, and $\|\vec{v}\| = \sqrt{6^2 + 3^2 + 2^2} = 7$. Thus,

$$\theta = \arccos\left(-\frac{4}{21} \right) \approx 1.762 \text{ rad} \approx 100.98°.$$

Orthogonal vectors

We want to extend the idea of *perpendicular* to more than 2 dimensions.

Definition. *Vectors $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> if $\vec{u} \cdot \vec{v} = 0$.*

Example. *Let $\vec{u} = \langle 3, -2, \rangle$ and $\vec{v} = \langle 4, 6 \rangle$. Then $\vec{u} \cdot \vec{v} = 3(4) + (-2)(6) = 0$.*
　　So $\vec{u}$ and $\vec{v}$ are orthogonal.

Example. *Similarly, $\vec{0}$ and any other vector are orthogonal, since $\vec{0} \cdot \vec{u} = 0(u_1) + 0(u_2) + 0(u_3) = 0.$*

Properties of the dot product

The dot product satisfies the following properties (listed on page 721 of the textbook):

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

2. $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$

3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

4. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

5. $\vec{0} \cdot \vec{u} = 0$

The textbook proves 1 & 3. Here's a proof of 4.
We have $\vec{u} \cdot \vec{u} = u_1(u_1) + u_2(u_2) + u_3(u_3) = u_1^2 + u_2^2 + u_3^2 = \|\vec{u}\|^2.$

Vector Projections

We can project one vector onto another vector.
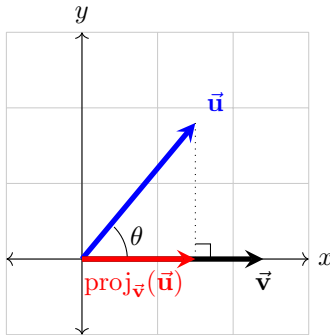
How do we do this algebraically? We want to find the vector $\text{proj}_{\vec{v}}(\vec{u})$. A vector is two pieces of information: A direction and a length. The direction is the same as the direction of $\vec{v}$. The length (from the picture) is the hypotenuse of the triangle times $\cos(\theta)$. Thus we have

$$\text{proj}_{\vec{v}}(\vec{u}) = \big(\|\vec{u}\| \cos(\theta)\big)\big(\text{unit vector in direction of } \vec{v}\big) = \big(\|\vec{u}\| \cos(\theta)\big)\left(\frac{\vec{v}}{\|\vec{v}\|}\right) = \left(\frac{\|\vec{u}\|(\vec{u} \cdot \vec{v})}{\|\vec{u}\|\|\vec{v}\|}\right)\left(\frac{\vec{v}}{\|\vec{v}\|}\right).$$

Canceling the common factor of $\|\vec{u}\|$, we get

$$\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v}.$$

Definition. *Let $\vec{u}$ and $\vec{v}$ be nonzero vectors. Then <u>the projection of $\vec{u}$ onto $\vec{v}$</u> is*

$$proj_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v}.$$

Note that $\text{proj}_{\vec{v}}(\vec{u}) \neq \text{proj}_{\vec{u}}(\vec{v})$. For example, in the picture above, $\text{proj}_{\vec{u}}(\vec{v})$ would go in the direction of $\vec{u}$, so they're related but not equal.

The number $\|\vec{u}\| \cos(\theta)\left(= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}\right)$ is called the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

Example. *Let $\vec{u} = 6\vec{i} + 3\vec{j} + 2\vec{k}$ and $\vec{v} = \vec{i} - 2\vec{j} - 2\vec{k}$. Then*

$$proj_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{6 - 6 - 4}{1 + 4 + 4}\right)\langle 1, -2, -2\rangle = -\frac{4}{9}\langle 1, -2, -2,\rangle = \langle -\frac{4}{9}, \frac{8}{9}, \frac{8}{9}\rangle.$$

Links

Robert Vandermolen's 12.3 slides:
    https://docs.google.com/presentation/d/1Vw6WT3sHAfts0ZrzP-5yjX37eaXP5Gk49jqckDkdXPI/edit#
slide=id.g1e7a467069_0_0

4 §12.4 The Cross Product

We want to define another vector product. Whereas the dot product gave us a number, we now want a vector product that gives us a vector.

To find a new vector, we need a length and a direction.

If we have any two nonzero, non-parallel vectors, those two vectors determine a plane. Then we can find a unit vector $\vec{\mathbf{n}}$ orthogonal to the plane using the right-hand rule. (i.e., it points in the direction your right thumb points when your fingers curve from $\vec{\mathbf{u}}$ to $\vec{\mathbf{v}}$.)

$\vec{\mathbf{n}}$ is a direction for our new vector, so we need a length. Calculate the area of the parallelogram formed by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, and this is the length of the new vector.
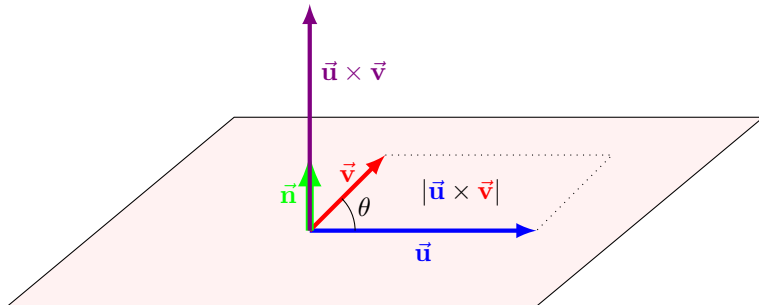
Definition. *The* <u>*cross product of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$*</u>, *denoted $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$, is the vector*

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = (\|\vec{\boldsymbol{u}}\|\|\vec{\boldsymbol{v}}\| \sin(\theta))\vec{\boldsymbol{n}}.$$

In practice, there's an easier way to calculate the cross product. Let's investigate the cross products of the standard unit vectors.

$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}} \qquad\qquad\qquad\qquad\qquad \vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}}$$
$$\vec{\mathbf{j}} \times \vec{\mathbf{k}} = \vec{\mathbf{i}} \qquad\qquad\qquad\qquad\qquad \vec{\mathbf{k}} \times \vec{\mathbf{j}} = -\vec{\mathbf{i}}$$
$$\vec{\mathbf{k}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \qquad\qquad\qquad\qquad\qquad \vec{\mathbf{i}} \times \vec{\mathbf{k}} = -\vec{\mathbf{j}}$$

We also have the property that $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$, i.e., that the cross product distributes across vector addition. We can use these facts to find a formula for the cross product.

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then we have

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_1\vec{\mathbf{i}} + u_2\vec{\mathbf{j}} + u_3\vec{\mathbf{k}}) \times (v_1\vec{\mathbf{i}} + v_2\vec{\mathbf{j}} + v_3\vec{\mathbf{k}})$$

$$= u_1v_1(\vec{\mathbf{i}}{\times}\vec{\mathbf{i}}) + u_1v_2(\vec{\mathbf{i}}{\times}\vec{\mathbf{j}}) + u_1v_3(\vec{\mathbf{i}}{\times}\vec{\mathbf{k}}) + u_2v_1(\vec{\mathbf{j}}{\times}\vec{\mathbf{i}}) + u_2v_2(\vec{\mathbf{j}}{\times}\vec{\mathbf{j}}) + u_2v_3(\vec{\mathbf{j}}{\times}\vec{\mathbf{k}}) + u_3v_1(\vec{\mathbf{k}}{\times}\vec{\mathbf{i}}) + u_3v_2(\vec{\mathbf{k}}{\times}\vec{\mathbf{j}}) + u_3v_3(\vec{\mathbf{k}}{\times}\vec{\mathbf{k}})$$

$$= \quad 0 \quad + u_1v_2(\vec{\mathbf{i}} \times \vec{\mathbf{j}}) + u_1v_3(\vec{\mathbf{i}} \times \vec{\mathbf{k}}) + u_2v_1(\vec{\mathbf{j}} \times \vec{\mathbf{i}}) + \quad 0 \quad + u_2v_3(\vec{\mathbf{j}} \times \vec{\mathbf{k}}) + u_3v_1(\vec{\mathbf{k}} \times \vec{\mathbf{i}}) + u_3v_2(\vec{\mathbf{k}} \times \vec{\mathbf{j}}) + \quad 0$$

$$= u_1v_2(\vec{\mathbf{k}}) + u_1v_3(-\vec{\mathbf{j}}) + u_2v_1(-\vec{\mathbf{k}}) + u_2v_3(\vec{\mathbf{i}}) + u_3v_1(\vec{\mathbf{j}}) + u_3v_2(-\vec{\mathbf{i}})$$

$$= (u_2v_3 - u_3v_2)\vec{\mathbf{i}} - (u_1v_3 - u_3v_1)\vec{\mathbf{j}} + (u_1v_2 - u_2v_1)\vec{\mathbf{k}}.$$

The components of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ can be recognized as determinants.

Determinants

Given a 2x2 matrix, we can calculate its determinant as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Given a 3x3 matrix, we can calculate its determinant as follows (beware of the minus before a_2):

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = a_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - a_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + a_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

In this language, we can write the cross product as the symbolic determinant

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example. *Let $\vec{u} = \langle 2, 1, 1 \rangle$ and $\vec{v} = \langle -4, 3, 1 \rangle$. Then we have*

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \vec{\mathbf{i}} \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - \vec{\mathbf{j}} \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + \vec{\mathbf{k}} \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = \vec{\mathbf{i}}(1 - 3) - \vec{\mathbf{j}}(2 + 4) + \vec{\mathbf{k}}(6 + 4) = -2\vec{\mathbf{i}} - 6\vec{\mathbf{j}} + 10\vec{\mathbf{k}}$$

So we have three equivalent ways of describing the cross product geometrically.

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \sin(\theta))\vec{\mathbf{n}} = (u_2 v_3 - u_3 v_2)\vec{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\vec{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\vec{\mathbf{k}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Properties of the Cross Product

The cross product satisfies several useful properties, which are given in the textbook at page 726.

1. $(r\vec{\mathbf{u}}) \times (s\vec{\mathbf{v}}) = (rs)(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$

2. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$

3. $\vec{\mathbf{v}} \times \vec{\mathbf{u}} = -(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$

4. $(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \times \vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{u}} + \vec{\mathbf{w}} \times \vec{\mathbf{u}}$

5. $\vec{\mathbf{0}} \times \vec{\mathbf{u}} = 0$

6. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\vec{\mathbf{w}}$
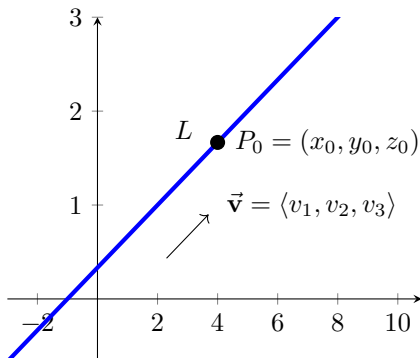
Links

Robert Vandermolen's 12.4 slides:
   `https://docs.google.com/presentation/d/1Vw6WT3sHAfts0ZrzP-5yjX37eaXP5Gk49jqckDkdXPI/edit#`
`slide=id.g22161d383c_2_25`

5 §12.5 Lines and Planes in Space

Lines

In the plane (2D), we needed a point and a slope to define a line. In space (3D), we need a *point* and a *direction vector* parallel to the line.

Let $P_0 = (x_0, y_0, z_0)$ be a fixed point on a line L which goes in the direction of the vector $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then the point (x, y, z) is on the line L if and only if (x, y, z) is in the direction of $\vec{\mathbf{v}}$ from P_0. In the language of vectors, this means

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle v_1, v_2, v_3 \rangle$$

where t is some real number, i.e., you can get to the point (x, y, z) by first going to (x_0, y_0, z_0) and then moving some distance in the direction of $\langle v_1, v_2, v_3 \rangle$.

But this goes in both directions. If you have a vector $\langle x, y, z \rangle$ that satisfies the equation above, then the point (x, y, z) is on the line L. Thus we can define a line parametrically as a vector equation.

Definition. *Let $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the vector equation of a line through $P_0 = (x_0, y_0, z_0)$ in the direction of $\vec{v}$ is*

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty.$$

There are many equivalent ways to write the equation of a line.

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}} = \langle x_0, y_0, z_0 \rangle + t \langle v_1, v_2, v_3 \rangle = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle.$$

Taking the last option, we can express a line as three parametric equations, where each variable is a function of t:

$$x(t) = x_0 + tv_1, \quad y(t) = y_0 + tv_2, \quad z(t) = z_0 + tv_3.$$

Example. *Find the equation of the line L passing through the points $P_1 = (-3, 2, -3)$ and $P_2 = (1, -1, 4)$.*

To find the equation for L, we must first find a direction vector that L is parallel to. The vector from P_1 to P_2 works fine. We calculate $\vec{\mathbf{v}} = \langle 1 - (-3), -1 - 2, 4 - (-3) \rangle = \langle 4, -3, 7 \rangle$.
Thus the vector equation for L is

$$\vec{\mathbf{r}}(t) = \langle 1, -1, 4 \rangle + t \langle 4, -3, 7 \rangle$$

. Notice that this is the same line as if we had chosen P_1 as our initial point. The equation

$$\vec{\mathbf{r}}(s) = \langle -3, 2, -3 \rangle + s \langle 4, -3, 7 \rangle$$

also describes L, just with different s or t values for each point.
If we wanted parametric equations for L, they would be $x = 1 + 4t, y = -1 - 3t, z = 4 + 7t$. The equations $x = -3 + 4s, y = 2 - 3s, z = -3 + 7s$ would also work.

Distance from a point to a line

To find the distance from a point to a line, we only need any point that lies on the line. Consider the following diagram. We want to know the distance from S to L, and we know the point P is on L. How can we use P to find this distance?
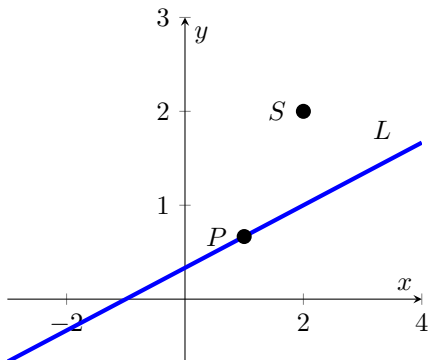
First, let's connect P and S and then indicate the distance (on the graph) that we're interested in. We will also need the direction vector $\vec{v}$ of L.
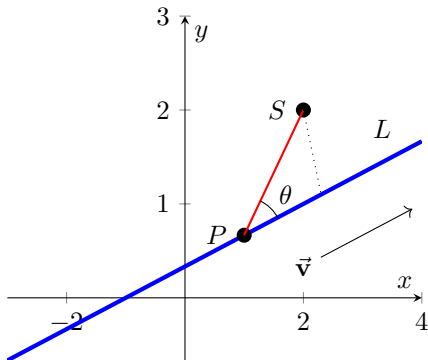
Using right-triangle trigonometry, we know that the length of the dotted line (the quantity we're interested in) is the length of the hypotenuse times $\sin(\theta)$. Thus the quantity we're interested in is

$$\|\vec{\mathbf{PS}}\| \sin(\theta).$$

We have seen $\sin(\theta)$ appear in the previous section in the formula for cross products:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \sin(\theta)) \vec{\mathbf{n}}$$

Thus,

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \sin(\theta).$$

If we take $\vec{\mathbf{u}} = \vec{\mathbf{PS}}$, then the distance we're interested in is

$$\|\vec{\mathbf{PS}}\| \sin(\theta) = \frac{\|\vec{\mathbf{PS}}\| \|\vec{\mathbf{v}}\| \sin(\theta)}{\|\vec{\mathbf{v}}\|} = \frac{\|\vec{\mathbf{PS}} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}$$

Note that this formula can only be used in three dimensions, since it has a cross product in it.

Example. *Find the distance from the point $(2, -3, 1)$ to the line $\vec{r}(t) = \langle 2t, 4 + 3t, 8 - t \rangle$.*

In this case, $S = (2, -3, 1)$ and $P = (0, 4, 8)$. Thus $\vec{\mathbf{PS}} = \langle 2, -7, -7 \rangle$. We also have $\vec{v} = \langle 2, 3, -1 \rangle$. This means

$$\vec{\mathbf{PS}} \times \vec{v} = \langle (7 + 21), -(-2 + 14), (6 + 14) \rangle = \langle 28, -12, 20 \rangle.$$

Thus the distance from S to the line is

$$\frac{\sqrt{28^2 + (-12)^2 + 20^2}}{\sqrt{2^2 + 3^2 + (-1)^2}} = \frac{\sqrt{1328}}{\sqrt{14}} = \sqrt{\frac{664}{7}}$$

Cont.

Planes

We can almost think about planes as the 3D version of 2D lines. When everything is said and done, we'll realize that every plane can be described by an equation where x, y and z appear to the first power (without being multiplied together). This is similar to lines in 2D, which can be described in general (as long as the line isn't vertical) by the equation $y = mx + b$.

Whereas we need a point and a direction to define a 3D line, we need a point and a tilt to define a plane. We will describe the tilt of a plane by its *normal vector*.

Definition. *A vector is <u>normal</u> to a plane if it is orthogonal (perpendicular) to the plane.*
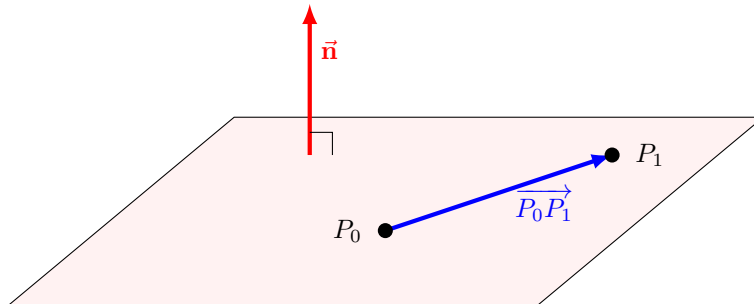
Every possible tilt of a plane gives us a different normal vector, so we can use the normal vector to keep track of the tilt.

We will take $P_0 = (x_0, y_0, z_0)$ to be a fixed point in the plane and $P_1 = (x, y, z)$ to be any other point in the plane.
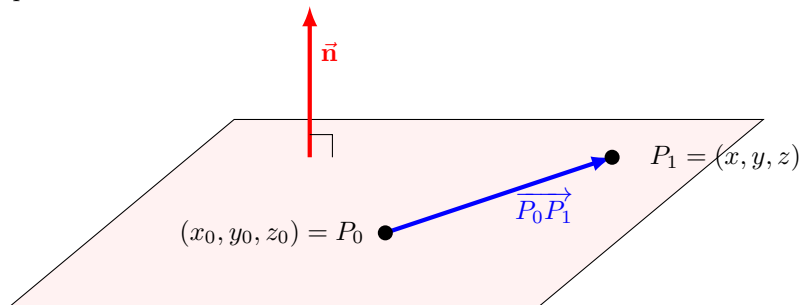
What does it mean for P_1 to be a point in the plane? That would mean that $\overrightarrow{P_0P_1}$ is a vector in the plane. But if $\vec{n}$ is normal to the plane, that would mean that $\overrightarrow{P_0P_1}$ is orthogonal to $\vec{n}$. Thus we have the defining equation of a plane: $\vec{n} \cdot \overrightarrow{P_0P_1} = 0$.

Definition. *Let $\vec{n} = \langle A, B, C \rangle$ be a normal vector to a plane containing the point $P_0 = (x_0, y_0, z_0)$. Then the equation of the plane (where $P_1 = (x, y, z)$) is*

$$\vec{n} \cdot \overrightarrow{P_0P_1} = 0$$

or

$$\langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0, \text{ the vector equation of a plane}$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \text{ the coordinate form of a plane}$$

or

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0 \text{ the coordinate form simplified.}$$

Example. *Find the equation of the plane through $(-3, 0, 7)$ with normal vector $\vec{n} = 5\vec{i} + 2\vec{j} - \vec{k}$.*

The vector equation is $\langle 5, 2, -1 \rangle \cdot \langle x + 3, y - 0, z - 7 \rangle = 0$. We can simplify this to its coordinate form $5(x + 3) + 2(y) - (z - 7) = 0$, or $5x + 2y - z = -22$.

Question. *What is the normal vector of the xy-plane? What is the normal vector of the plane $3x - 6y = z + 8$?*

$\vec{\mathbf{k}}$ and $\langle 3, -6, -1 \rangle$ (or $\langle -3, 6, 1 \rangle$).

Using the properties of cross products, we can do more complicated examples.

Example. *Find the equation of the plane through $R = (0, 0, 1)$, $S = (2, 0, 0)$, and $T = (0, 3, 0)$.*

We need to find a normal vector. To do that, we will find the cross product of two vectors in the plane. Two such vectors are $\vec{RS} = \langle 2, 0, -1 \rangle$ and $\vec{RT} = \langle 0, 3, -1 \rangle$. Then we have the cross product

$$\vec{RS} \times \vec{RT} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \vec{\mathbf{k}} = \langle 3, 2, 6 \rangle.$$

This vector is orthogonal to two vectors in the plane, so it must be orthogonal to every vector in the plane, and hence is a normal vector for the plane. Taking $\vec{\mathbf{n}} = \langle 3, 2, 6 \rangle$, we have the equation of the plane is

$$\langle 3, 2, 6, \rangle \cdot \langle x - 0, y - 0, z - 1 \rangle = 0, \text{ or } 3x + 2y + 6z = 6$$

.

Whenever two planes meet, they intersect at a line. We can find the equation of the line of intersection using cross products.

Example. *Find the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.*

To find the equation of a line, we need a point on the line and a direction vector. The direction vector is in both planes, so it must be orthogonal to the normal vectors of each plane. This means that if we take the cross product of the normal vectors of each plane, we will get a vector parallel to the line of intersection (since it will be a vector in both planes). We have

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -6 & -2 \\ 1 & -2 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 3 & -6 \\ 2 & 1 \end{vmatrix} \vec{\mathbf{k}} = \langle 14, 2, 15 \rangle.$$

This is the direction vector for the line of intersection. Next we need a point on the line. Since both planes pass through the xy-plane, we can find where they intersect there. In the xy-plane, $z = 0$, so the planes have the form

$$3x - 6y = 15 \text{ and } 2x + y = 5.$$

Adding 6 times the second equation to the first equation gives $15x + 0y = 45$, or $x = 3$. Then substituting this value into the second equation gives $6 + y = 5$, or $y = -1$. Thus the planes intersect in the xy-plane at $x = 3$ and $y = -1$. This means the point $(3, -1, 0)$ is on the line of intersection.

Combining the information we have found, the line of intersection is given by the vector equation $\vec{\mathbf{r}}(t) = \langle 3, -1, 0 \rangle + t \langle 14, 2, 15 \rangle$, with parametric equations $x = 3 + 14t, y = -1 + 2t, z = 15t$.

We can also find points of intersection between a plane and a line by simple substitution of the line equation into the plane equation.

Example. *Find the point of intersection between the line $\vec{\mathbf{r}}(t) = \langle \frac{8}{3} + 2t, -2t, 1 + t \rangle$ and the plane $3x + 2y + 6z = 6$.*
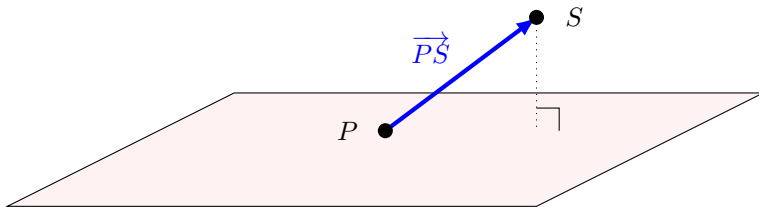
The line is described by the parametric equations $x = \frac{8}{3} + 2t, y = -2, z = 1 + t$. Substituting this into the equation for the plane gives

$$3(\frac{8}{3} + 2t) + 2(-2t) + 6(1 + t) = 6 \Rightarrow 8 + 6t - 4t + 6 + 6t = 6 \Rightarrow 8t = -8 \Rightarrow t = -1.$$

Thus the point of intersection is $(\frac{8}{3} + 2(-1), -2(-1), 1 - 1) = (\frac{2}{3}, 2, 0)$.

Distance from a point to a plane

Suppose that P is a point on a plane and S is any other point. We have the following picture.


We are not interested in the length of $\overrightarrow{PS}$. What we want is the distance from S to the plane, i.e., the length of the dotted line.

Notice that the dotted line is parallel to the normal vector of the plane. If we insert the normal vector into the picture originating at P, then we can use projection vectors to find the length of the dotted line.
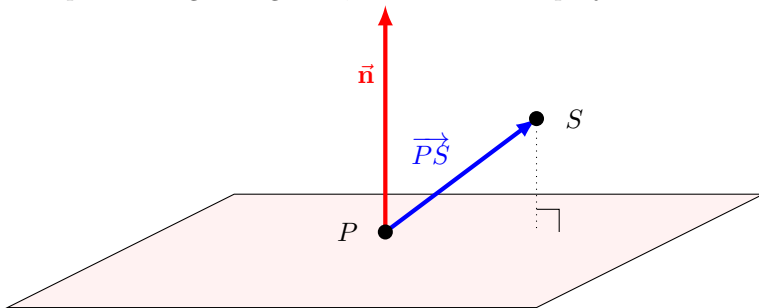
If we project $\overrightarrow{PS}$ onto $\vec{\mathbf{n}}$, we get the following vector, whose length is the same as the length of the dotted line:
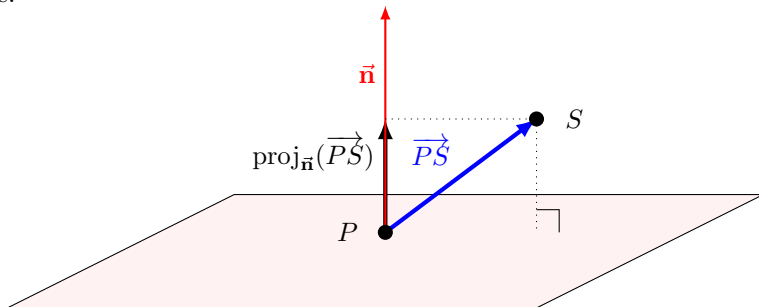
Recall that the length of $\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}})$ is $\dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}$. Thus, the distance we're interested in is

$$\|\text{proj}_{\vec{\mathbf{n}}}(\overrightarrow{PS})\| = \frac{\overrightarrow{PS} \cdot \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|}.$$

Definition. *Let S be any point and P be a point in a plane with normal vector $\vec{\boldsymbol{n}}$. Then the distance from S to the plane is*

$$\frac{\overrightarrow{PS} \cdot \vec{\boldsymbol{n}}}{\|\vec{\boldsymbol{n}}\|}.$$

Example. *Let $S = (1, 1, 3)$. Find the distance from S to the plane $3x + 2y + 6z = 6$.*

The normal vector for $3x + 2y + 6z = 6$ is $\vec{\mathbf{n}} = \langle 3, 2, 6 \rangle$. By inspection, we see that $P = (0, 3, 0)$ is a point on the plane. So $\overrightarrow{PS} = \langle 1, -2, 3 \rangle$. Thus the distance from S to the plane is $\frac{\langle 1, -2, 3 \rangle \cdot \langle 3, 2, 6 \rangle}{\|\langle 3, 2, 6 \rangle\|} = \frac{3 - 4 + 18}{\sqrt{9 + 4 + 36}} = \frac{17}{7}$.
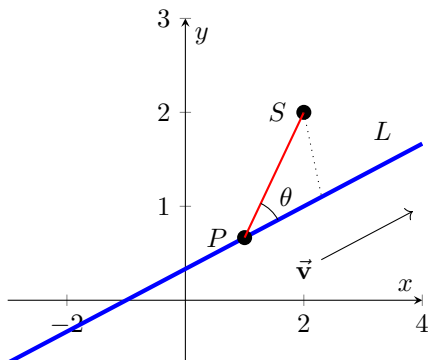
Links

Robert Vandermolen's 12.5 slides:

`https://docs.google.com/presentation/d/1PbKAu9bEMq2A5aV2VGG86C1uHGBIqZJgSg1uaonTKOY/edit#slide=id.p`

12.5 Distance activity (next page):

Work together in your group on the problems below. You may be asked to put your work on the board.

Name:_____

1. This problem is about finding the distance between a point and a line.

Here's the information we know:

We want the distance from a point S to the line L (indicated by the dashed line).

We have a point P on the line.

We have a direction vector $\vec{v}$ for the line L.

(a) Draw the vector $\overrightarrow{PS}$ in the graph.

(b) Mark the angle between $\overrightarrow{PS}$ and the line L.

(c) Use right-triangle trigonometry to find the length of the dashed line. (Recall SOHCAHTOA.)

We have $\sin(\theta) = \frac{length}{\|\overrightarrow{PS}\|}$, so solving for length gives $length = \|\overrightarrow{PS}\| \sin(\theta)$.

(d) What is the length of the vector $\overrightarrow{PS} \times \vec{v}$? (You may have to check your notes.)

$\|\overrightarrow{PS} \times \vec{v}\| = \|\overrightarrow{PS}\|\|\vec{v}\| \sin(\theta)$ (this is a formula from the original definition of the cross product.)

(e) What do you have to divide your answer in problem d by to get the distance in problem c?

$$\text{Distance in problem c} = \frac{\|\overrightarrow{PS} \times \vec{v}\|}{\|\vec{v}\|}$$

(f) Write down a formula for the distance from S to the L without using the angle between $\overrightarrow{PS}$ and L.
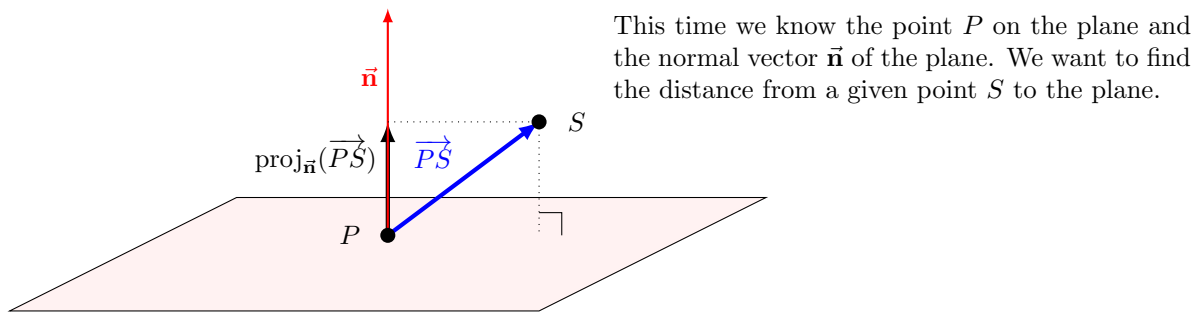
$\text{Distance} = \text{Distance in problem c} = \frac{\|\overrightarrow{PS} \times \vec{v}\|}{\|\vec{v}\|}$

(g) Find the distance from $S = (1, -3, 2)$ to the line $\vec{r}(t) = \langle 2 - 4t, 5 + 2t, 3 + 4t \rangle$.

We have $\vec{v} = \langle -4, 2, 4 \rangle$. A point on the line is $P = (2, 5, 3)$ (This can be found by reading off the numbers without a t in each component, or equivalently by taking $t = 0$.) $\overrightarrow{PS} = \langle -1, -8, -1 \rangle$. A calculation shows that $\overrightarrow{PS} \times \vec{v} = \langle -30, 8, -34 \rangle$. Thus we have

$$\text{distance} = \frac{\sqrt{30^2 + 8^2 + 34^2}}{\sqrt{16 + 4 + 16}} = \frac{\sqrt{2120}}{6}.$$

2. This problem is about finding the distance between a point and a plane.

This time we know the point P on the plane and the normal vector $\vec{\mathbf{n}}$ of the plane. We want to find the distance from a given point S to the plane.

(a) Draw the vector $\overrightarrow{PS}$ in the graph.

(b) Mark the angle between $\overrightarrow{PS}$ and $\vec{\mathbf{n}}$.

(c) What projection vector has length equal to length of the dotted line?

$\text{proj}_{\vec{\mathbf{n}}}(\overrightarrow{PS})$

(d) What is the length of this projection vector? (You may have to check your notes.)

The length is the scalar component of $\overrightarrow{PS}$ in the direction of $\vec{\mathbf{n}}$, or $\frac{\overrightarrow{PS} \cdot \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|}$. (You should convince yourself that this is a number).

(e) Write down a formula for the distance from S to the plane without using the angle between $\overrightarrow{PS}$ and $\vec{\mathbf{n}}$.

$\frac{\overrightarrow{PS} \cdot \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|}$. Technically we should take the absolute value of the dot product so that we actually get a positive distance. $\frac{|\overrightarrow{PS} \cdot \vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}$

(f) Find the distance from $S = (2, -3, 5)$ to the plane with normal vector $\vec{\mathbf{n}} = \langle 3, -1, 6 \rangle$ containing the point $P = (4, 0, 3)$.

We have $\overrightarrow{PS} = \langle -2, -3, 2 \rangle$. So $\overrightarrow{PS} \cdot \vec{\mathbf{n}} = -6 + 3 + 12 = 9$. We have $\|\vec{\mathbf{n}}\| = \sqrt{9 + 1 + 36} = \sqrt{46}$. Thus the distance we're looking for is
$$\frac{9}{\sqrt{46}}.$$

3. Find the distance from the point $(0, 0, 12)$ to the line $x = 4t, y = -2t, z = 2t$.

Plugging $t = 0$ into the line, we have $P = (0, 0, 0)$ is on the line. Taking $S = (0, 0, 12)$, we have $\overrightarrow{PS} = \langle 0, 0, 12 \rangle$. Taking the coefficients on t in the line, we have $\vec{v} = \langle 4, -2, 2 \rangle$. Then

$$\frac{\|\overrightarrow{PS} \times \vec{v}\|}{\|\vec{v}\|} = \frac{\|\langle 24, 48, 0 \rangle\|}{\sqrt{16 + 4 + 4}} = 24 \frac{\|\langle 1, 2, 0 \rangle\|}{\sqrt{24}} = \sqrt{24}(\sqrt{1 + 4}) = 2\sqrt{30}.$$

4. Find the distance from the point $(3, -1, 4)$ to the line $\vec{r}(t) = \langle 4 - t, 3 + 2t, -5 + 3t \rangle$.

Plugging $t = 0$ into the line, we have $P = (4, 3, -5)$ is on the line. Taking $S = (3, -1, 4)$, we have $\overrightarrow{PS} = \langle -1, -4, 9 \rangle$. Taking the coefficients on t in the line, we have $\vec{v} = \langle -1, 2, 3 \rangle$. Then

$$\frac{\|\overrightarrow{PS} \times \vec{v}\|}{\|\vec{v}\|} = \frac{\|\langle -30, -6, -6 \rangle\|}{\sqrt{1 + 4 + 9}} = 6 \frac{\|\langle -5, -1, -1 \rangle\|}{\sqrt{13}} = \frac{6(\sqrt{25 + 1 + 1})}{\sqrt{13}} = \frac{6\sqrt{27}}{\sqrt{13}}.$$

5. Find the distance from the point $(2, -3, 4)$ to the plane $x + 2y + 2z = 13$.

The normal vector $\vec{n}$ for the plane is $\langle 1, 2, 2 \rangle$. We can find a point in the plane by finding a point (x, y, z) that satisfies the equation of the plane. For example, $(13, 0, 0)$ and $(5, 4, 4)$ are on the plane (we only need one of these). Taking $P = (13, 0, 0)$, we have $\overrightarrow{PS} = \langle -11, -3, 4 \rangle$. Then

$$\frac{\overrightarrow{PS} \cdot \vec{n}}{\|\vec{n}\|} = \frac{-11 - 6 + 8}{\sqrt{1 + 4 + 4}} = -9/3 = -3$$

The distance is the absolute value of this quantity, or 3.

6. Find the distance from the point $(0, -1, 0)$ to the plane $2x + y + 2z = 4$.

We have $\vec{n} = \langle 2, 1, 2 \rangle$. A point in the plane is $P = (1, 0, 1)$. We have $\overrightarrow{PS} = \langle -1, -1, -1 \rangle$. Then

$$\frac{|\overrightarrow{PS} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|-2 + 1 - 2|}{\sqrt{4 + 1 + 4}} = \frac{3}{3} = 1.$$

7. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

Since these planes have the same normal vectors, they are parallel planes. This means that it doesn't matter what point you pick to find the distance between: Every point on the first plane is the same distance from the second plane. So we'll pick any point on the first plane: $S = (1, 0, 0)$ is on $x + 2y + 6z = 1$.

Then P is any point on the plane $x + 2y + 6z = 10$. Let's take $P = (2, 1, 1)$. Then $\overrightarrow{PS} = \langle -1, -1, -1 \rangle$. Using our distance to plane formula, we have

$$\frac{|\overrightarrow{PS} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|-1 - 2 - 6|}{\sqrt{1 + 4 + 36}} = \frac{9}{\sqrt{41}}$$

8. Find the distance from the line $x = 2 + t, y = 1 + t, z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

In this problem, the line and the plane don't intersect. We can see this because the line's direction vector is orthogonal to the plane's normal vector: $1(1) + 1(2) + (-1/2)(6) = 0$. So either the line is in the plane entirely or the two objects don't intersect. Checking $2 + 2(1) + 6(-1/2) \neq 10$, we know that they don't intersect. So let $S = (2, 1, -1/2)$ and $P = (2, 1, 1)$. Then we can use our distance to a plane formula to find that S is

$$\frac{|\overrightarrow{PS} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|\langle 0, 0, -3/2 \rangle \cdot \langle 1, 2, 6 \rangle|}{\sqrt{1 + 4 + 36}} = \frac{9}{\sqrt{41}}$$

away from the plane. Thus the line is the same distance away.

9. The equation $\vec{n} \cdot \overrightarrow{P_0 P_1} = 0$ represents the plane through P_0 normal to $\vec{n}$. What set does the inequality $\vec{n} \cdot \overrightarrow{P_0 P_1} > 0$ represent?

The inequality $\vec{n} \cdot \overrightarrow{P_0 P_1} > 0$ describes all the points above the plane $\vec{n} \cdot \overrightarrow{P_0 P_1} = 0$, where *above* means in the direction of $\vec{n}$. For example, the inequality $z > 0$ is all the points in the direction of $\langle 0, 0, 1 \rangle$ from the xy-plane.

You can see this by replacing the dot product with another definition of the dot product: $\|\overrightarrow{P_0 P_1}\| \|\vec{n}\| \cos(\theta) > 0$ means that $\cos(\theta) > 0$ and thus $0 < \theta < \pi/2$, i.e., the angle is acute, so the vector formed by joining P_0 to P_1 points in the direction of $\vec{n}$, which means (x, y, z) lies in the direction of $\vec{n}$ from the plane.

The End.

6 §12.6 Cylinders and Quadric surfaces

We'll return to this section once we're past Chapters 12 and 13. We'll follow Robert Vandermolen's slides fairly exclusively for this.

Robert Vandermolen's 12.6 Slides
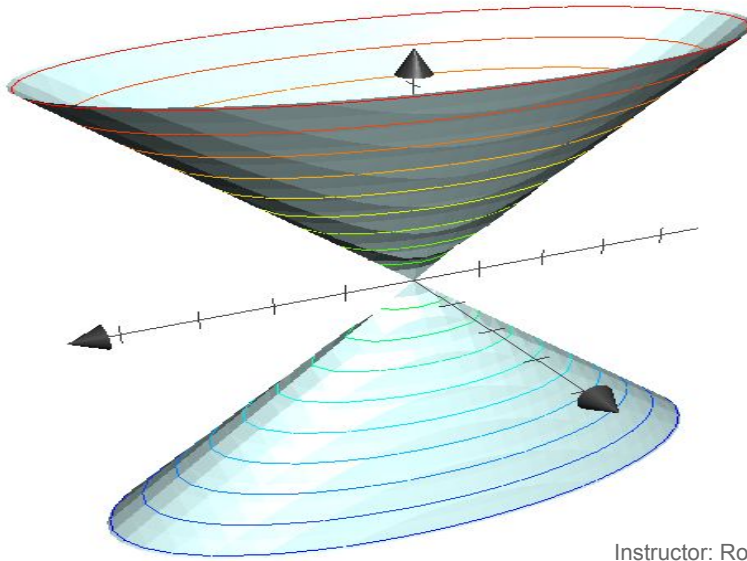
Instructor: Robert Vandermolen
(12.6)

Contour Graphing!

Two calculus classes later we are really good at graphing in 2-dimensions, so let's use this to our advantage...

Instead of drawing the whole picture, we will first just draw slices of the picture

Let's try it with the following surface first:

$$x^2 + y^2 + z^2 = 9$$

Contour Graphing!

We begin by choosing different values for z and graphing the function's
x and y-coordinates

$$x^2 + y^2 + z^2 = 9$$

Why didn't I pick values of z past 3 or below -3?

z=0

z=1

z=-1

z=2

z=-2
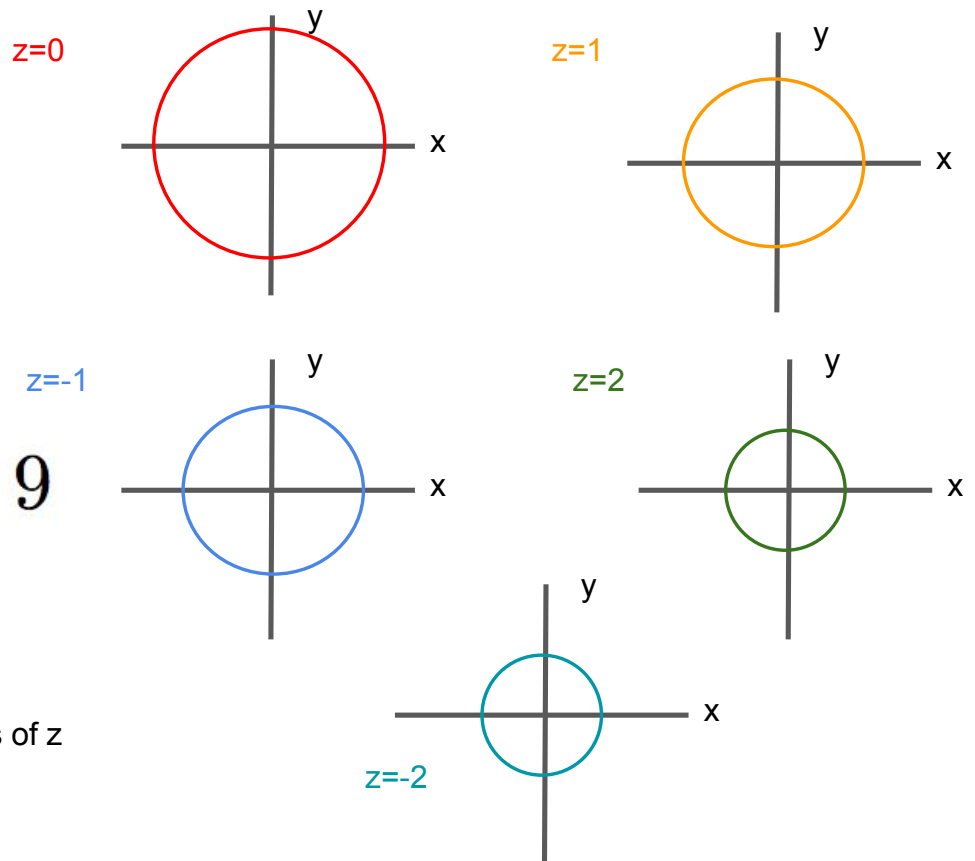
Contour Graphing!

Now we put the contours together!

$$x^2 + y^2 + z^2 = 9$$

z+

z=2

z=1

y+

z=0

z=-1

x+

z=-2

Looky a sphere!!
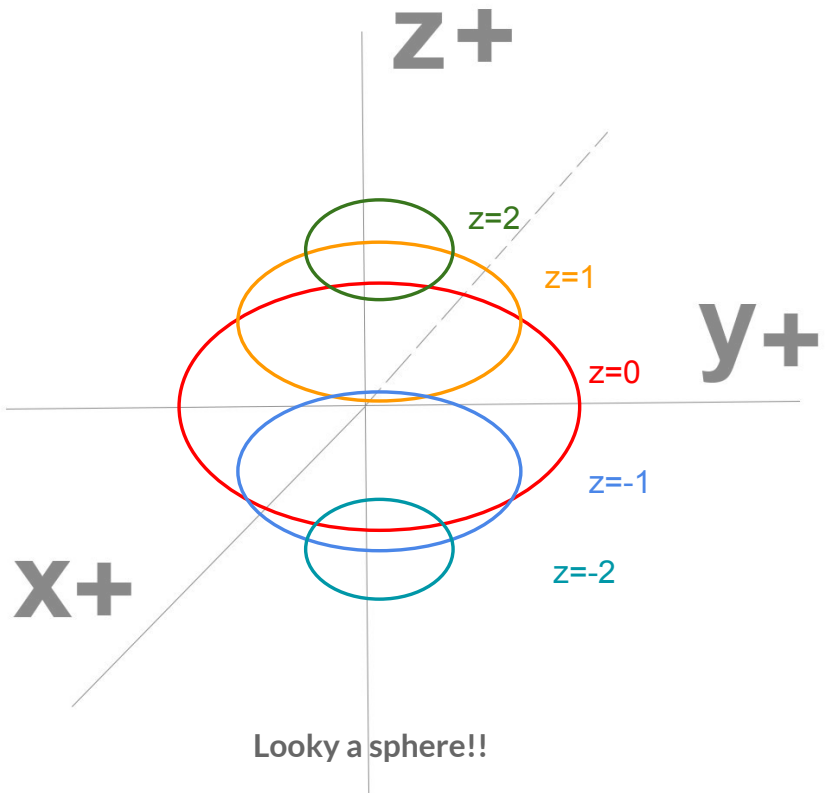
Contour Graphing!

We could have done the same thing by picking different values for x, and graphing the remaining coordinates

$$x^2 + y^2 + z^2 = 9$$

x=0

z

y

x=1

z

y

x=-1

z

y

x=3

z

y

x=-3

z

y
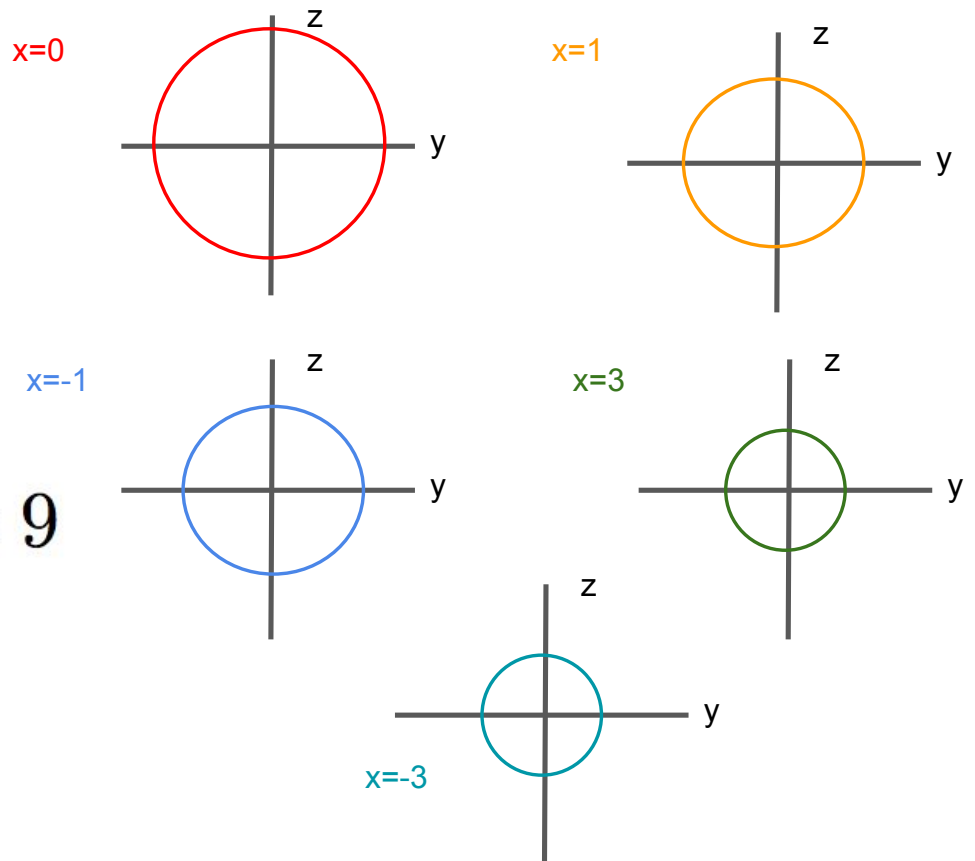
Contour Graphing!

Again putting the contours together we get:

$$x^2 + y^2 + z^2 = 9$$

Looky a sphere!!
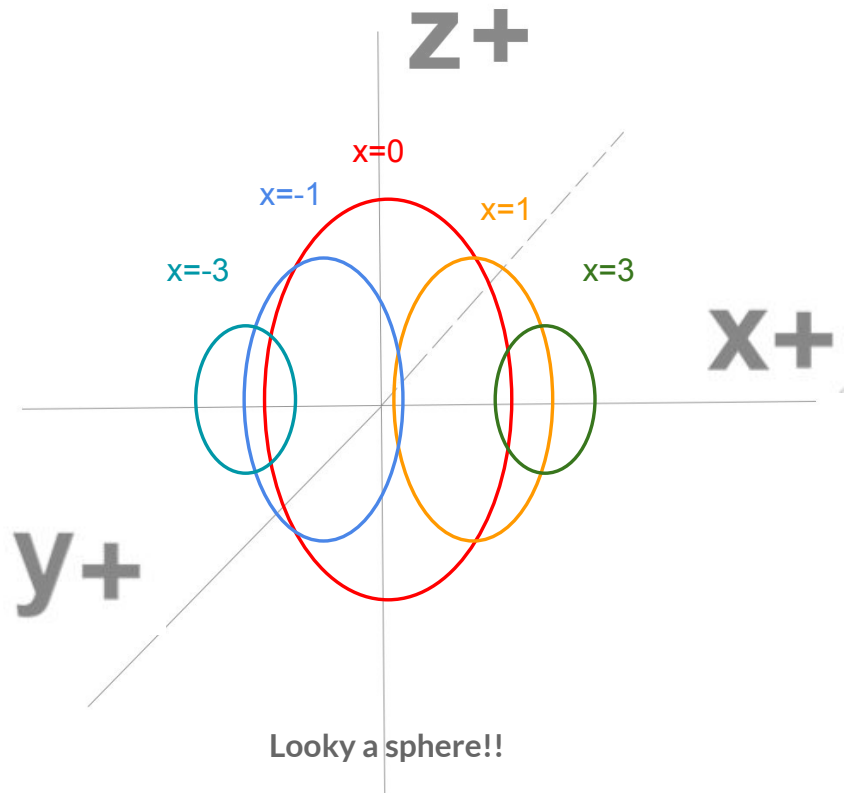
Contour Graphing!

For this function when we choose different values of x the function doesn't change

$$z^2 + y^2 = 9$$

Look they are all the same!
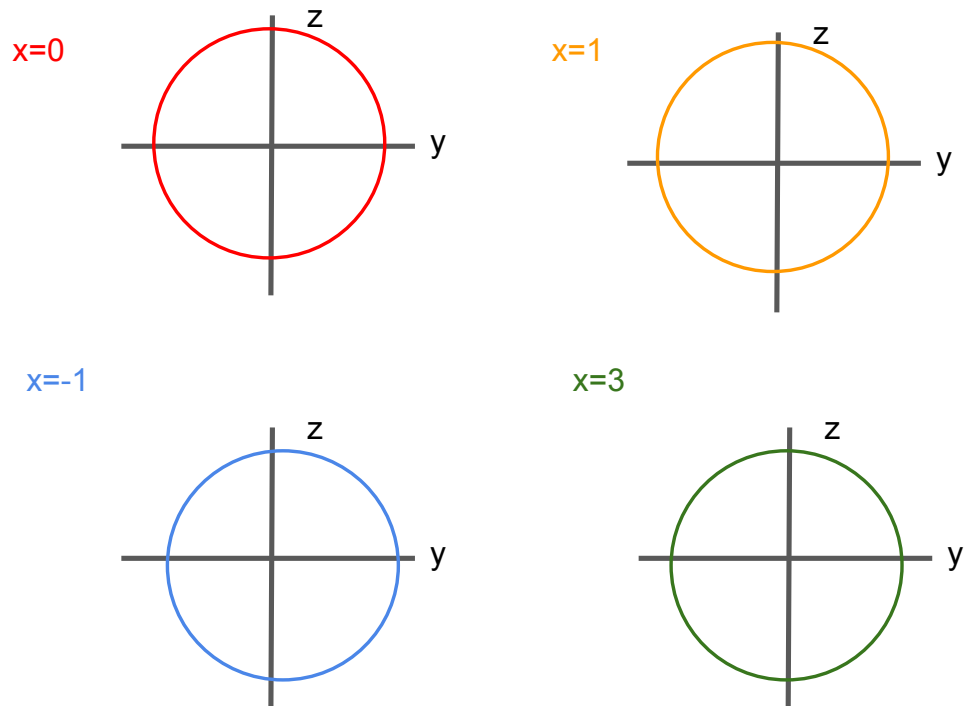
Contour Graphing!

When we put together the contours the picture reveals itself

$$z^2 + y^2 = 9$$

x=-1   x=0   x=1   x=3

It is a cylinder!
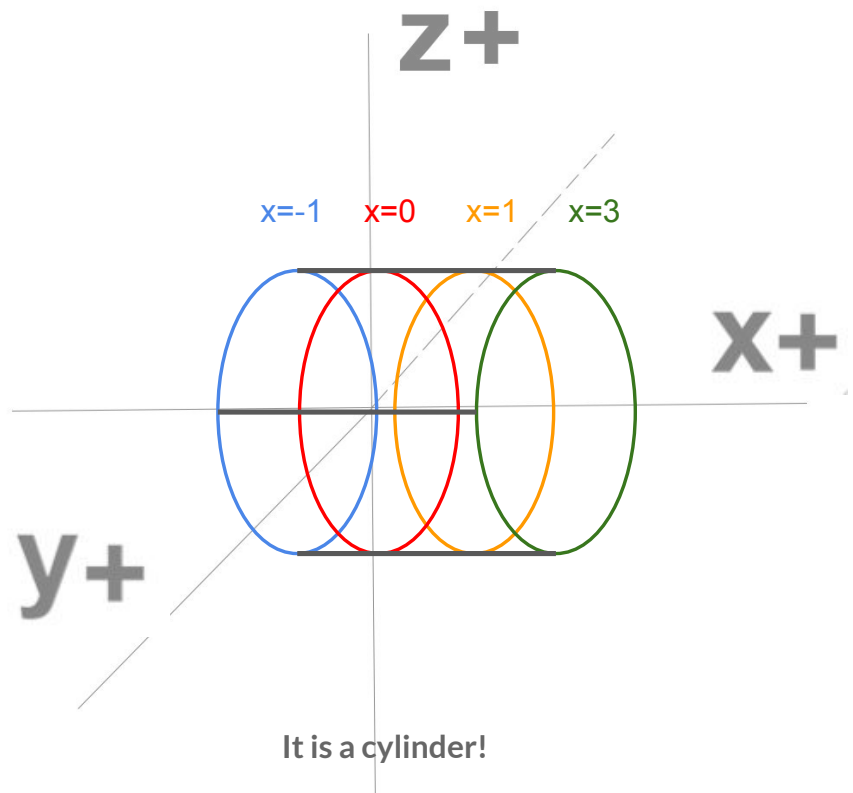
Quadratic Surfaces!

A basic **Quadratic Surface** has the form:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where A,B,C,D,E,F,G,H,I,J are numbers and quite often a lot of them are zero!

ELLIPTIC PARABOLOID!

Let's do the same thing.
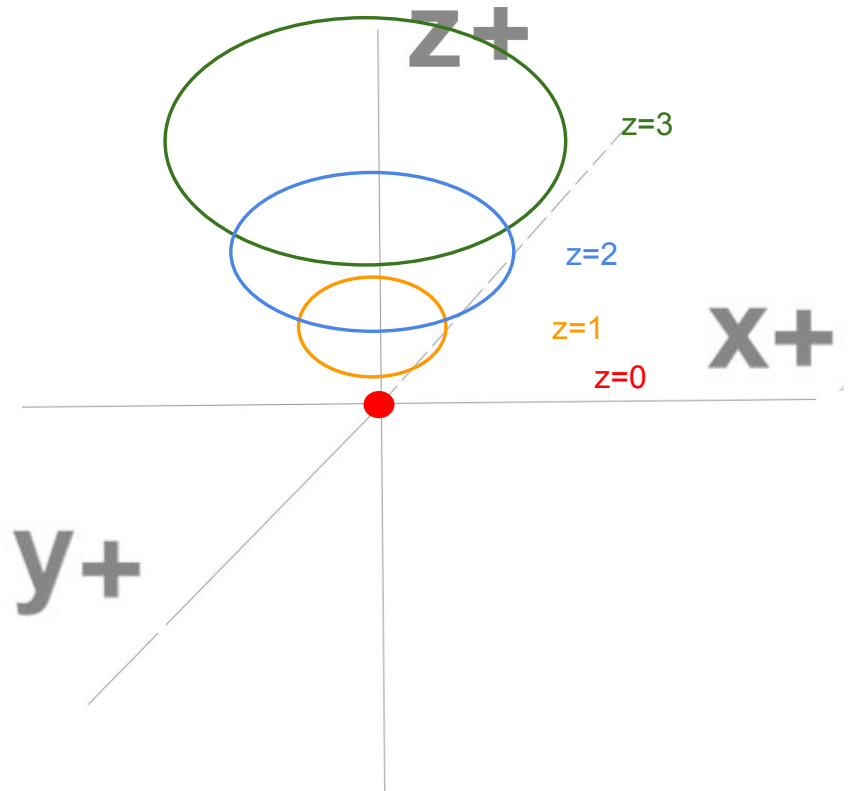
We will start with choosing values for z

$$z = x^2 + y^2$$

z=0

z=1

z=2

z=3
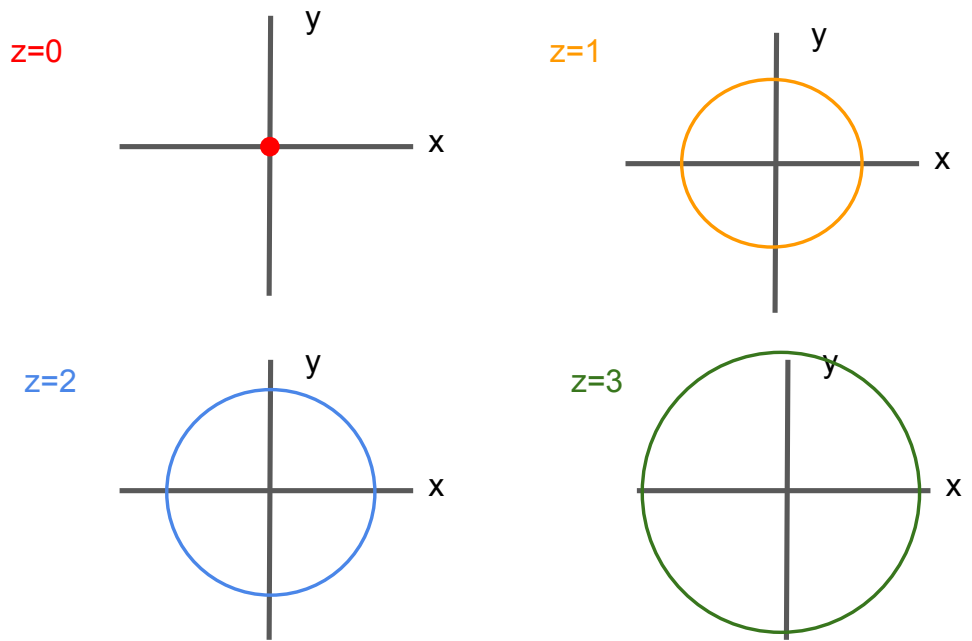
Why did I not choose negative values?

ELLIPTIC PARABOLOID!

Now we put the contours together!
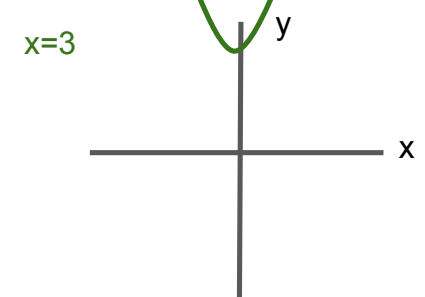
$$z = x^2 + y^2$$

z+
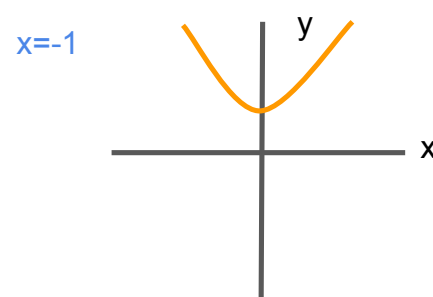
z=3

z=2

z=1

z=0

X+

y+

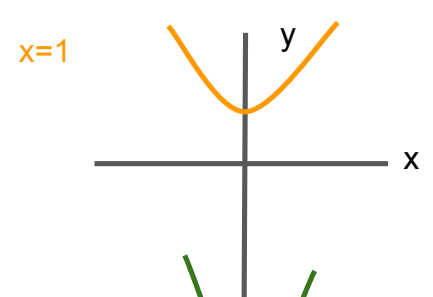# ELLIPTIC PARABOLOID!

Let's do the same thing.

What if we did it this time choosing values for x

$$z = x^2 + y^2$$

x=0

x=1

x=-1

x=3

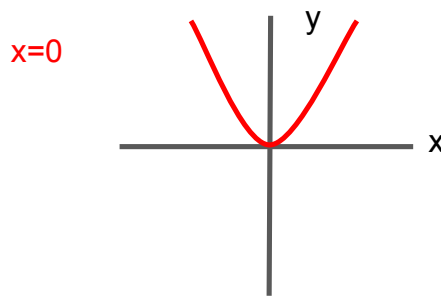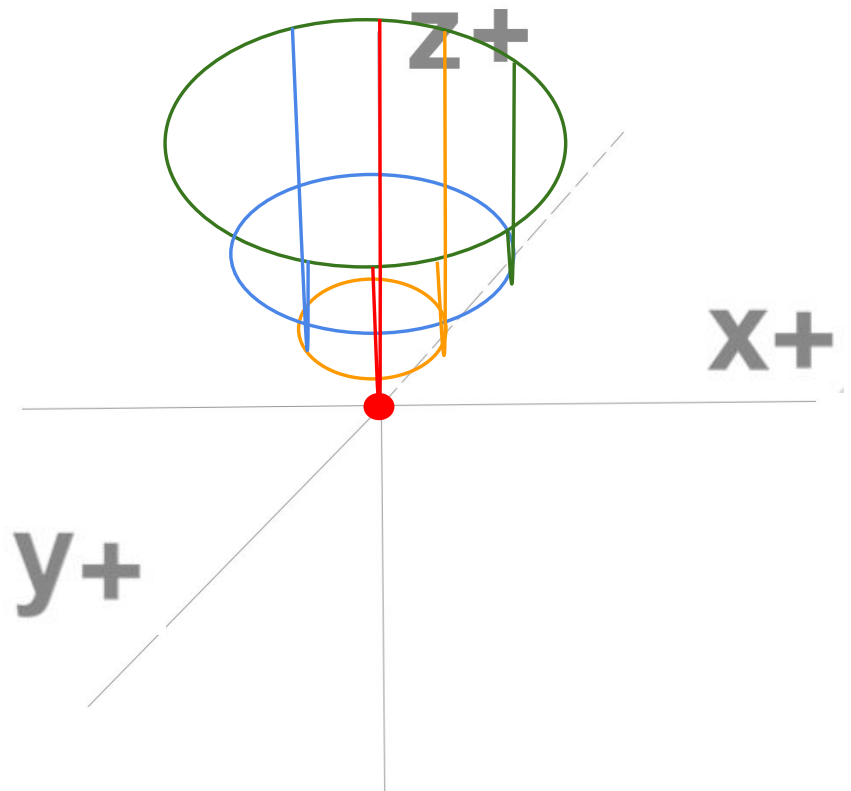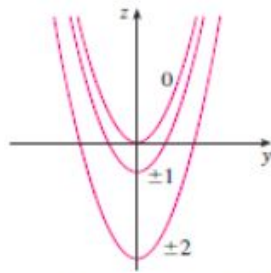# ELLIPTIC PARABOLOID!

Now we put the contours together!

$$z = x^2 + y^2$$

Hyperbolic Paraboloid!

Let's do the same thing.

$$z = y^2 - x^2$$
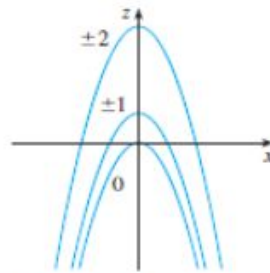
Choosing values for x   Choosing values for y   Choosing values for z
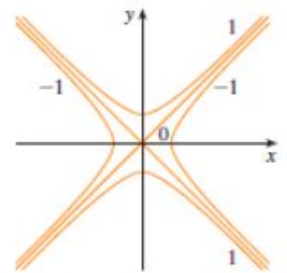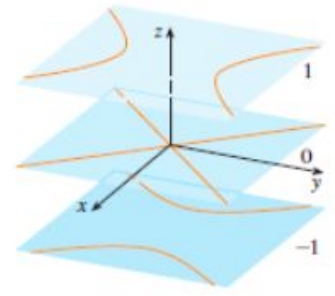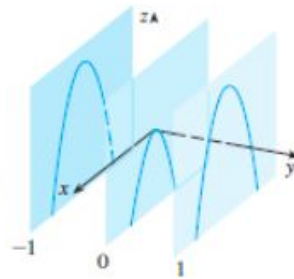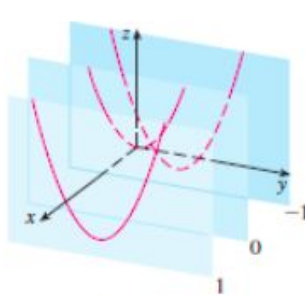
Hyperbolic Paraboloid!

Putting it together!

$$z = y^2 - x^2$$

HYPERBOLA

Part of the hyperbola $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$
in the plane $z = c$

Part of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
in the plane $z = -c$
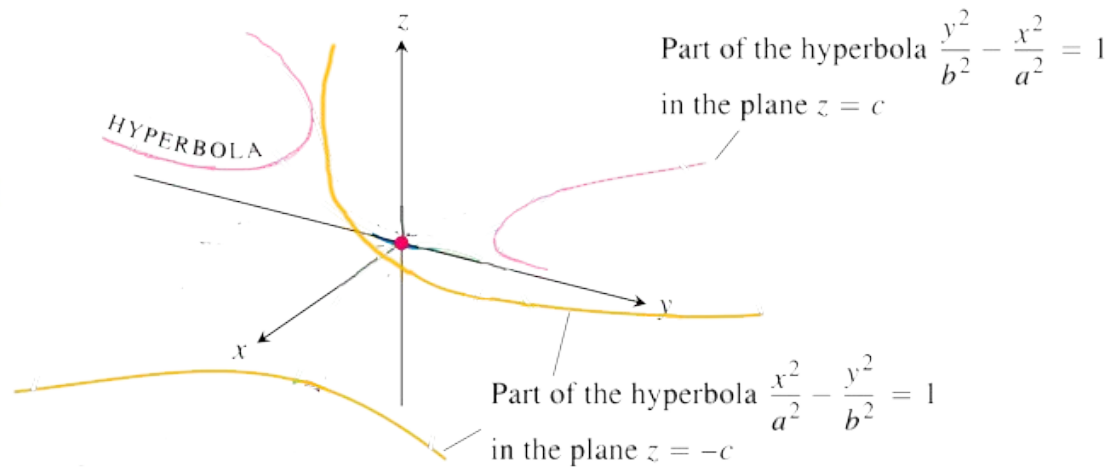
HYPERBOLIC PARABOLOID!

Putting it together!

The parabola $z = \frac{c}{b^2} y^2$ in the yz-plane

Part of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ in the plane $z = c$

$$z = y^2 - x^2$$

HYPERBOLA

PARABOLA

Saddle point

PARABOLA

Part of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the plane $z = -c$

The parabola $z = -\frac{c}{a^2} x^2$ in the xz-plane
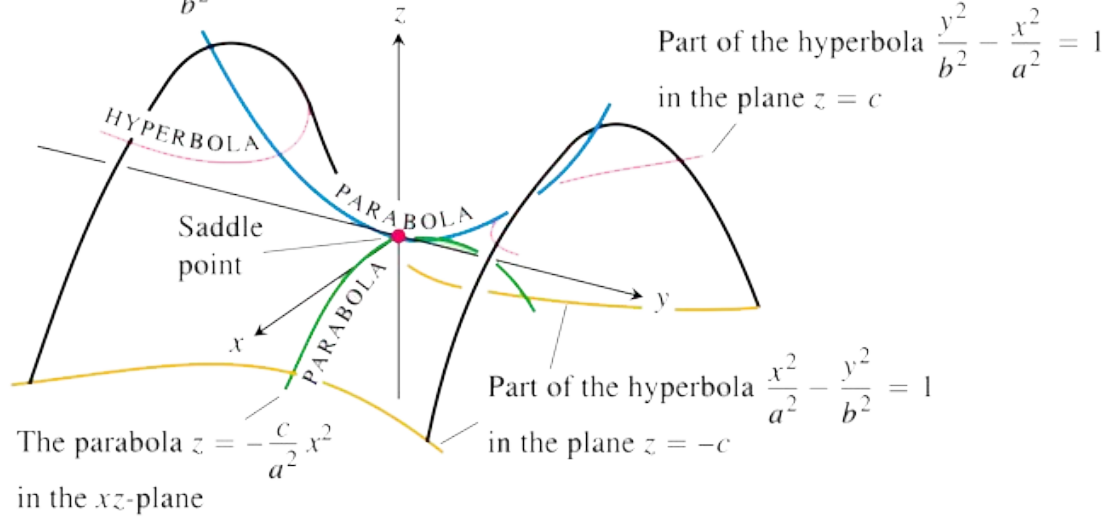
HYPERBOLIC PARABOLOID!

Putting it together!

$$z = y^2 - x$$
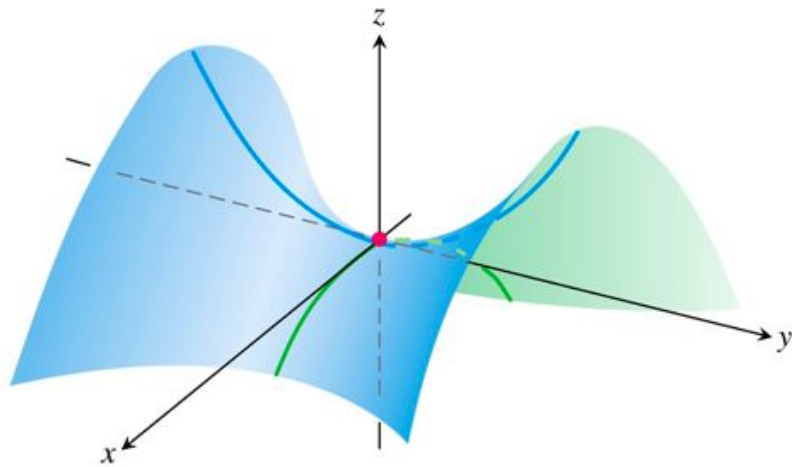
Now You Try!

Use this method to draw the following curve:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

It is called an
Elliptic Cone

Answer!
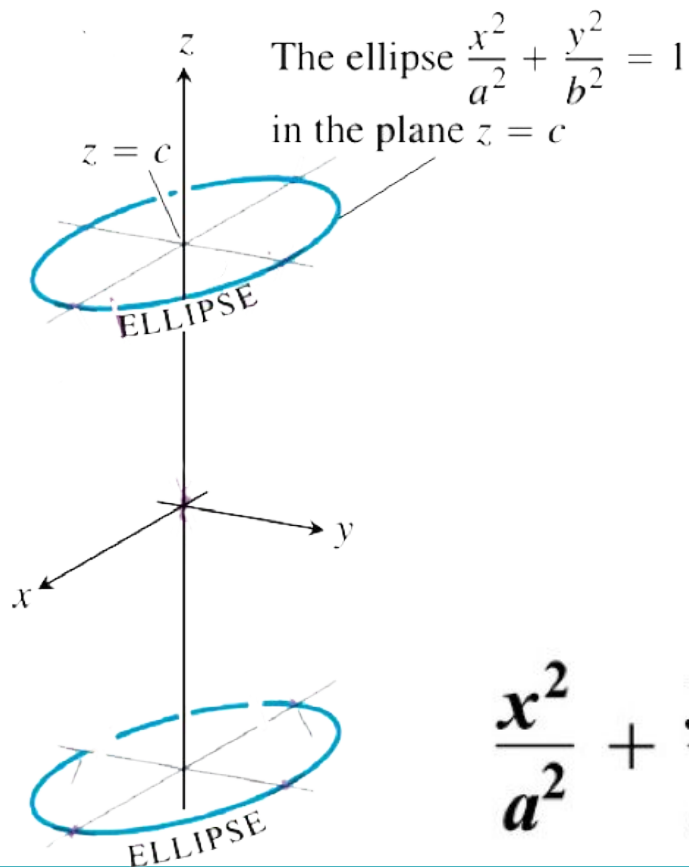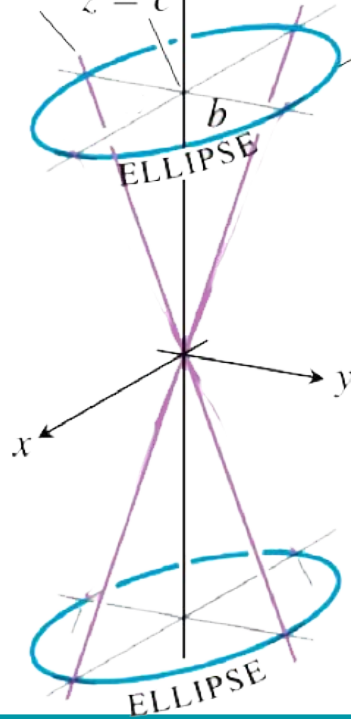
The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
in the plane $z = c$

$z = c$

ELLIPSE

z

y

x

ELLIPSE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

The line $z = -\dfrac{c}{b} y$

in the yz-plane

$z = c$

The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

in the plane $z = c$

b

ELLIPSE

z

y

x

ELLIPSE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

The line $z = -\dfrac{c}{b} y$

in the yz-plane

$z = c$

The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

in the plane $z = c$

a b

ELLIPSE

The line $z = \dfrac{c}{a} x$

in the xz-plane

z

y

x

ELLIPSE
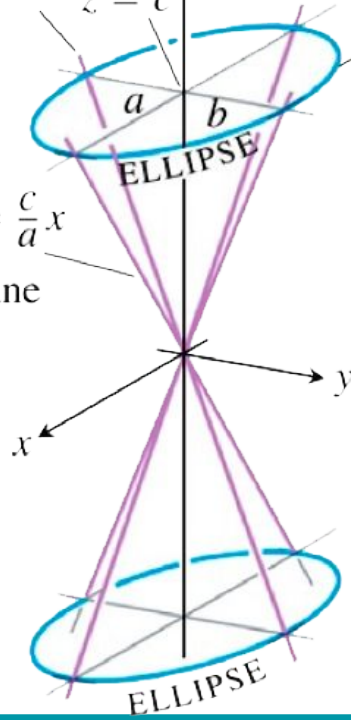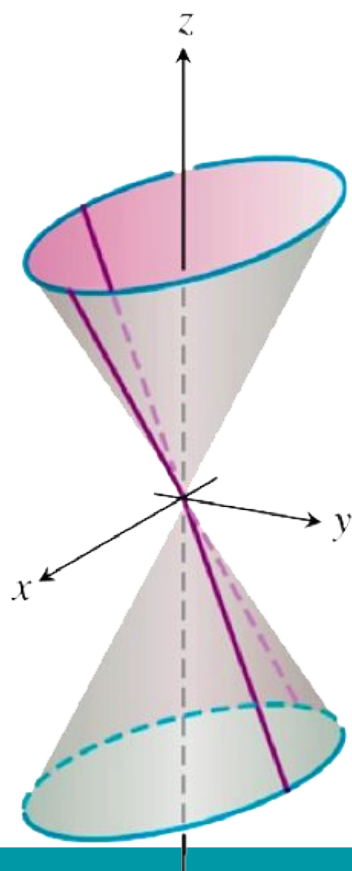
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Links

Robert Vandermolen's 12.6 slides:
    `https://docs.google.com/presentation/d/19O0SIsn4YMgyKJtBSag3Uy64uBxVKlp35jZJN2-Po0c/edit#slide=id.p`

7 §13.1 Curves in space and their tangents

There are many reasons to keep track of curves in space. For example, a curve could represent the path of a particle passing through a liquid or any other similar phenomenon. To keep track of such curves, we use vector-valued functions.

Definition. *A vector-valued function is a function*

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where f, g, and h are real-valued functions.

We have already seen an example of a space curve, lines.

Example. *The function $\vec{r}(t) = \langle 1 + t, -2t, 3 + 5t \rangle$ is a space curve (in this case, also a line).*

There are also more complex examples, such as the helix below.

Example. *$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$, graphed from $t = 0$ to $t = 2\pi$.*
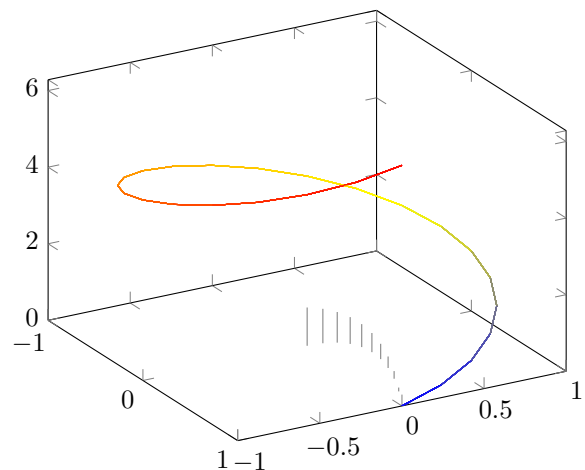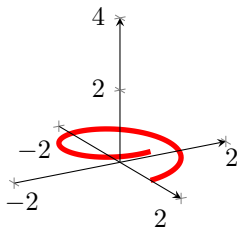
Limits

Limits of vector-valued functions are calculated in each component.

Definition. *Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then*

$$\lim_{t \to t_0} \vec{r}(t) = \left\langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right\rangle$$

and the limit exists if all three limits exist.

Example. *Let $\vec{r}(t) = \left\langle \dfrac{\sin(t)}{t}, e^{-t} + 5, \dfrac{t^2}{2t^2 - 4} \right\rangle$. What is $\lim_{t \to 0} \vec{r}(t)$?*

Using our knowledge of limits from Calculus 1, we get $\lim_{t \to 0} \vec{r}(t) = \langle 1, 6, 0 \rangle$.

Continuity

Continuity is also defined in each component.

Definition. *Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then $\vec{r}(t)$ is continuous at $t = t_0$ if*

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0),$$

that is, it is both the case that the limit exists and is equal to $\vec{r}(t_0)$.

To check continuity, one simply checks continuity in each component.

Example. *Consider the helix,* $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. *Each function* $\cos(t), \sin(t),$ *and* t *is continuous on* $\mathbb{R}$. *Thus* $\vec{r}(t)$ *is continuous.*

The example $\vec{r}(t) = \left\langle \dfrac{\sin(t)}{t}, e^{-t} + 5, \dfrac{t^2}{2t^2 - 4} \right\rangle$ would not be continuous at $t = 0$, however, since $\vec{r}(0)$ is not defined.

Derivatives

We can also take derivatives of vector-valued functions. Once again, this simply happens in each component.

Definition. *Let* $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. $\vec{r}$ *is differentiable at* $t = t_0$ *if* f, g *and* h *are. In this case,*

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \left\langle \frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt} \right\rangle.$$

We can visualize $\frac{d\vec{r}}{dt}$ geometrically as the tangent vector to the space curve.
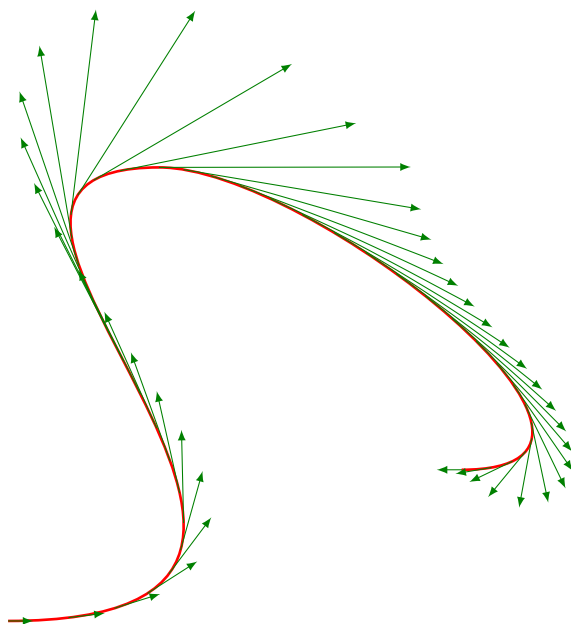
When we are thinking of $\vec{r}(t)$ as the path of a particle in space, the first derivative represents the velocity of the particle and the second derivative represents the acceleration. In this case, we use the notation

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \text{ and } \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}.$$

The quantity $\|\vec{v}(t)\|$ represents the speed of the particle at time t. Notice that this is a real-valued function.

Example. *Find* $\vec{v}(t), \|\vec{v}(t)\|,$ *and* $\vec{a}(t)$ *for the helix* $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

We have $\vec{v}(t) = \langle -\sin(t), \cos(t), 1 \rangle$ and $\vec{a}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$. We calculate

$$\|\vec{v}\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1)^2} = \sqrt{1 + 1} = \sqrt{2}.$$

If $\frac{d\vec{r}}{dt}$ is never 0, then the space curve has a well-defined direction at all points of the curve. Vector-valued functions with this property are *smooth*.

Definition. *A vector-valued function $\vec{r}(t)$ is smooth on the domain D if*

1. $\frac{d\vec{r}}{dt}$ *is continuous on D, and*

2. $\frac{d\vec{r}}{dt}$ *is never 0 on D.*

We simply say $\vec{r}(t)$ is smooth if $\vec{r}(t)$ is smooth on $\mathbb{R}$.

    For example, the helix is smooth.

Example. *Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Then $\frac{d\vec{r}}{dt} = \langle -\sin(t), \cos(t), 1 \rangle$. Since the z-direction of $\frac{d\vec{r}}{dt}$ is always 1, $\frac{d\vec{r}}{dt}$ is never the zero vector for any value of t. Thus $\vec{r}(t)$ is smooth.*

Differentiation Rules

The derivatives of vector-valued functions follow many of the familiar derivative rules from Calculus 1, and the product rule extends to both of the dot product and the cross product.

    Let $\vec{u}, \vec{v}$ be vector-valued functions. Let $\vec{C}$ be a constant vector. Let $k \in \mathbb{R}$ be a scalar, and let $f(t)$ be a differentiable real-valued function. Then the following rules hold.

1. $\frac{d}{dt}\left[\vec{C}\right] = \vec{0}$

2. $\frac{d}{dt}\left[k\vec{u}(t)\right] = c\frac{d\vec{u}}{dt}$

    $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = \frac{df}{dt}\vec{u}(t) + f(t)\frac{d\vec{u}}{dt}$

3. $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}$

5. $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \cdot \vec{v}(t) + \vec{u}(t) \cdot \frac{d\vec{v}}{dt}$

6. $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \times \vec{v}(t) + \vec{u}(t) \times \frac{d\vec{v}}{dt}$

7. $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\vec{u}'(f(t))$

Vector-valued functions of constant length

If a vector-valued function always has the same length, i.e., if it takes a path along a sphere centered at the origin, then we have the following fact: $\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$. This is effectively saying that the derivative vector $\frac{d\vec{r}}{dt}$ is tangent to the sphere at each point on the space curve's path, which makes sense if the function is always on the sphere. Here is an algebraic proof:

    If $\vec{r}(t)$ has constant length, then we have $\vec{r}(t) \cdot \vec{r}(t) = c^2$ for some length c. This means

$$\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{r}(t)\right] = 0 \Rightarrow \frac{d\vec{r}}{dt} \cdot \vec{r}(t) + \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0 \Rightarrow 2\vec{r}(t)\frac{d\vec{r}}{dt} = 0$$

which gives the desired result.

Links

Robert Vandermolen's 13.1 slides:
    `https://docs.google.com/presentation/d/1_ErYYu3YotlInL19AtinC32WESgYFfmsEJkYcoJfLwM/edit#slide=id.p`

8   §13.2 Integrals of Vector Functions

We can integrate vector functions componentwise just as we differentiated them in the previous section.

Definition. *Let $\vec{r}(t)$ be a vector function with $\overrightarrow{R}(t)$ an anti-derivative for $\vec{r}(t)$. Then*

$$\int \vec{r}(t)dt = \overrightarrow{R}(t) + \vec{C}$$

for some constant vector $\vec{C}$.

Notice that just like finding antiderivatives in Calculus 2, this is a function (in this case a vector function).

Example. *Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find $\int \vec{r}(t)dt$.*

The integral is $\langle \sin(t), -\cos(t), \frac{t^2}{2} \rangle + \langle c_1, c_2, c_3 \rangle$ where the c_i are constant real numbers.

We can also calculate definite integrals.

Definition. *If the components of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ are integrable over the interval $[a, b]$, then so is $\vec{r}$, and*

$$\int_a^b \vec{r}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$$

Notice that a definite integral gives us a vector in 3D, not a function.

Example.

$$\int_0^\pi (\cos(t)\vec{i} + \vec{j} - 2t\vec{k})dt = [\sin(t)]_0^\pi \vec{i} + [t]_0^\pi \vec{j} + [t^2]_0^\pi \vec{k} = (0-0)\vec{i} + (\pi - 0)\vec{k} + [\pi^2 - 0]\vec{k} = \langle 0, \pi, \pi^2 \rangle.$$

Links

Robert Vandermolen's 13.2 slides:
    `https://docs.google.com/presentation/d/1_ErYYu3YotlInL19AtinC32WESgYFfmsEJkYcoJfLwM/edit#`
`slide=id.p`

9   §13.3 Arclength in space

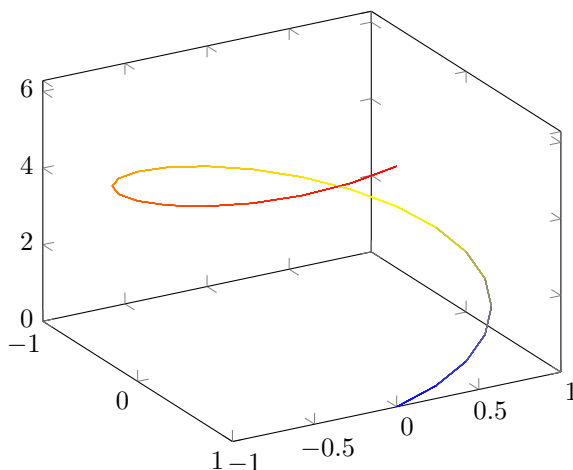We can find the distance a space curve travels between two points by integrating the distance formula.

Definition. *Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be smooth and let $a \le t \le b$. Then the length of $\vec{r}$ from $t = a$ to $t = b$ is*

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\,dt.$$

Example. *Find the length of the helix $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ from $t = 0$ to $t = 2\pi$.*

The length is

$$L = \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}\,dt = \int_0^{2\pi} \sqrt{1 + 1}\,dt = \sqrt{2}t\Big]_{t=0}^{t=2\pi} = 2\sqrt{2}\pi.$$

We can turn this formula into a function if we think of one of the bounds as a variable.

Definition. *The arclength parameter with base point t_0 is*

$$s(t) = \int_{t_0}^t \sqrt{\left(\frac{df}{d\tau}\right)^2 + \left(\frac{dg}{d\tau}\right)^2 + \left(\frac{dh}{d\tau}\right)^2}\,d\tau$$

Example. *Find the arclength parameter of the helix with base point 0.*

The arclength parameter is

$$s(t) = \int_0^t \sqrt{(-\sin(\tau))^2 + (\cos(\tau))^2 + 1^2}\,d\tau = \int_0^t \sqrt{1+1}\,d\tau = \sqrt{2}\tau\Big]_0^t = \sqrt{2}t.$$

In the event that the function $s(t)$ is invertible, i.e., if it can be solved for $t = t(s)$ as a function of s, then we can compose $\vec{r}(t)$ and $t(s)$ to get a formula representing where the space curve is after traveling a certain length from the base point.

Example. *In the example above, we found $s(t) = \sqrt{2}t$. Solving for t, we get a function $t(s) = \frac{s}{\sqrt{2}}$. Thus*

$$\vec{r}(t(s)) = (\cos(\frac{s}{\sqrt{2}}))\vec{i} + (\sin(\frac{s}{\sqrt{2}}))\vec{j} + \frac{s}{\sqrt{2}}\vec{k}$$

is the arclength parametrization for the helix.

The function $\vec{r}(t(s))$ allows us to plug in a distance traveled from the point t_0, which outputs the position in space the curve $\vec{r}(t)$ is after traveling that far.

Links

Robert Vandermolen's 13.3 slides:
  https://docs.google.com/presentation/d/12srwxuU-ecRSIezuA9kGSNdzopTR5IVTdxUKG2Gs7Ec/edit#slide=id.p

10 §13.3-13.5 Velocity and normal vectors of a curve

There are three important vectors associated with a space curve. These are the unit tangent vector $\overrightarrow{\mathbf{T}}$, the unit normal vector $\overrightarrow{\mathbf{N}}$, and the unit binormal vector $\overrightarrow{\mathbf{B}}$.

Definition. *Let $\vec{r}(t)$ be a smooth curve. Let $\vec{v}(t) = \frac{d\vec{r}}{dt}$. Then the unit tangent vector $\overrightarrow{\boldsymbol{T}}$ of $\vec{r}$ is*

$$\overrightarrow{\boldsymbol{T}} := \frac{\vec{v}}{\|\vec{v}\|}.$$
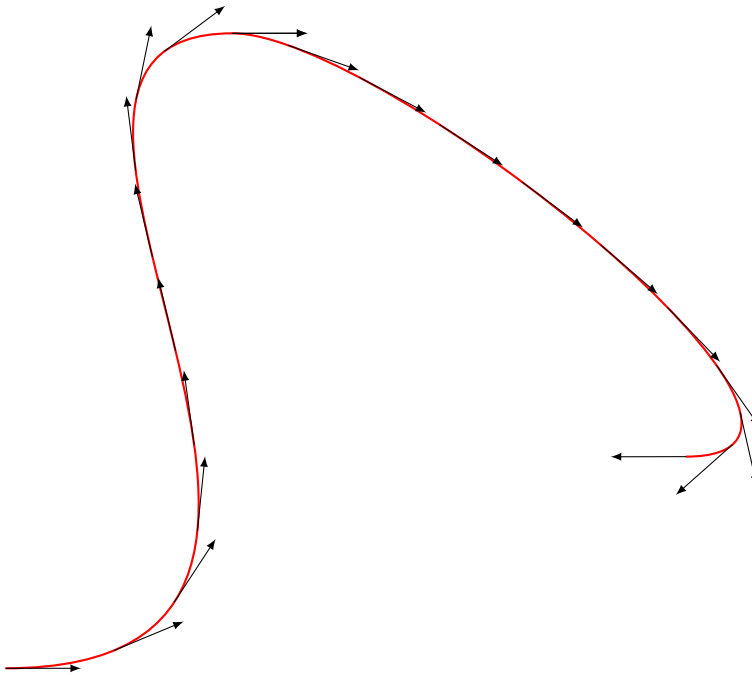
Example. *Let $\vec{r}(t) = (1 + 3\cos(t))\vec{i} + (3\sin(t))\vec{j} + t^2\vec{k}$. Find $\overrightarrow{\boldsymbol{T}}$.*

First we differentiate $\mathbf{r}(t)$ to find $\mathbf{v}(t)$, the velocity vector of the space curve.

$$\vec{\mathbf{v}}(t) = \langle -3\sin(t), 3\cos(t), 2t \rangle.$$

Next, we find the length of $\vec{\mathbf{v}}(t)$. Since $\vec{\mathbf{v}}(t)$ is a vector function, its length will be a scalar function.

$$\|\vec{\mathbf{v}}\| = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2 + (2t)^2} = \sqrt{9\sin^2(t) + 9\cos^2(t) + 4t^2} = \sqrt{9 + 4t^2}.$$

Then $\overrightarrow{\mathbf{T}}$ is $\vec{\mathbf{v}}$ divided by $\|\vec{\mathbf{v}}\|$.

$$\overrightarrow{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{\langle -3\sin(t), 3\cos(t), 2t \rangle}{\sqrt{9 + 4t^2}} = \left\langle \frac{-3\sin(t)}{\sqrt{9 + 4t^2}}, \frac{3\cos(t)}{\sqrt{9 + 4t^2}}, \frac{2t}{\sqrt{9 + 4t^2}} \right\rangle.$$

Notice that because the length of $\overrightarrow{\mathbf{T}}$ is fixed (it always has length 1), we know that $\overrightarrow{\mathbf{T}}$ is orthogonal to its derivative. Recall from section 13.1 that if $\|\vec{\mathbf{r}}(t)\| = c^2$ for some fixed c, then

$$\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t) \right] = 0 \Rightarrow \frac{d\vec{\mathbf{r}}}{dt} \cdot \vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t) \cdot \frac{d\vec{\mathbf{r}}}{dt} = 0 \Rightarrow 2\vec{\mathbf{r}}(t)\frac{d\vec{\mathbf{r}}}{dt} = 0$$

which means $\vec{\mathbf{r}}(t)$ and $\frac{d\vec{\mathbf{r}}}{dt}$ are orthogonal.

Cont.

In this case, we have that since $\overrightarrow{\mathbf{T}}$ is a fixed length, $\overrightarrow{\mathbf{T}}$ and $\frac{d\overrightarrow{\mathbf{T}}}{dt}$ are orthogonal.
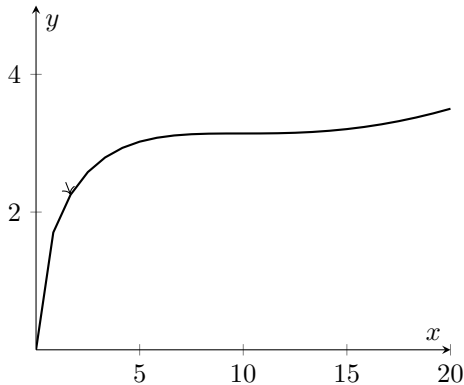
$\frac{d\overrightarrow{\mathbf{T}}}{dt}$ always points toward the 'inside' of the space curve, or in the direction that the curve is going.

We will define the unit normal vector in terms of $\frac{d\overrightarrow{\mathbf{T}}}{dt}$.

Definition. *The vector function*

$$\overrightarrow{\boldsymbol{N}} = \frac{\frac{d\overrightarrow{\boldsymbol{T}}}{dt}}{\|\frac{d\overrightarrow{\boldsymbol{T}}}{dt}\|}$$

is the (principal) unit normal vector of $\vec{r}(t)$, when it exists.

Notice that we always have to calculate $\overrightarrow{\mathbf{T}}$ (and its derivative) before we can find $\overrightarrow{\mathbf{N}}$.

Example. *Let $\vec{r}(t) = t\vec{i} + \ln(\cos(t))\vec{j}$, $-\pi/2 < t < \pi/2$. Find $\overrightarrow{\boldsymbol{N}}$.*

We have

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{i}} + \frac{-\sin(t)}{\cos(t)}\vec{\mathbf{j}}.$$

We must divide this by the length $\|\vec{\mathbf{v}}(t)\|$ to find $\overrightarrow{\mathbf{T}}$.

$$\|\vec{\mathbf{v}}(t)\| = \sqrt{1 + \frac{\sin^2(t)}{\cos^2(t)}} = \sqrt{1 + \tan^2(t)} = \sqrt{\sec^2(t)} = \sec(t) \text{ because } -\pi/2 < t < \pi/2.$$

Thus we have

$$\overrightarrow{\mathbf{T}} = \frac{1}{\sec(t)}\vec{\mathbf{i}} - \frac{\tan(t)}{\sec(t)}\vec{\mathbf{j}} = \cos(t)\vec{\mathbf{i}} - \sin(t)\vec{\mathbf{j}}.$$

Then to find $\overrightarrow{\mathbf{N}}$, we must differentiate $\overrightarrow{\mathbf{T}}$ and divide by its length.

$$\frac{d\overrightarrow{\mathbf{T}}}{dt} = -\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}$$

Since the length of $\frac{d\overrightarrow{\mathbf{T}}}{dt}$ is $\sqrt{\sin^t(t) + \cos^2(t)} = 1$, this is already a unit vector.

$$\overrightarrow{\mathbf{N}} = -\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}.$$

Now that we have $\overrightarrow{\mathbf{T}}$ and $\overrightarrow{\mathbf{N}}$, we can define the binormal vector.

Definition. *The binormal vector $\overrightarrow{\boldsymbol{B}}$ of a curve in space is*

$$\overrightarrow{\boldsymbol{B}} = \overrightarrow{\boldsymbol{T}} \times \overrightarrow{\boldsymbol{N}}.$$

Cont.

Notice that since $\|\overrightarrow{\mathbf{B}}\| = \|\overrightarrow{\mathbf{T}}\|\|\overrightarrow{\mathbf{N}}\|\sin(\pi/2) = 1$, the vector $\overrightarrow{\mathbf{B}}$ is already a unit vector by definition, so we don't need to divide by its length.

Example. *Take as above* $\overrightarrow{\boldsymbol{T}} = \langle\cos(t), -\sin(t), 0\rangle$ *and* $\overrightarrow{\boldsymbol{N}} = \langle-\sin(t), -\cos(t), 0\rangle$. *Find* $\overrightarrow{\boldsymbol{B}}$.

We have (using the determinant formula for cross products)

$$\overrightarrow{\mathbf{B}} = \overrightarrow{\mathbf{T}} \times \overrightarrow{\mathbf{N}} = \langle 0 - 0, 0 - 0, -\cos^2(t) - \sin^2(t)\rangle = \langle 0, 0, -1\rangle.$$

In this case $\overrightarrow{\mathbf{B}}$ is a fixed vector, but in most cases $\overrightarrow{\mathbf{B}}$ will be a vector-valued function just like $\overrightarrow{\mathbf{T}}$ and $\overrightarrow{\mathbf{N}}$.

The vectors $\overrightarrow{\mathbf{T}}, \overrightarrow{\mathbf{N}}$, and $\overrightarrow{\mathbf{B}}$ form a new set of mutually orthogonal vectors in space. This set is called the Frenet Frame (or the TNB Frame), and they are useful in physical applications.

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Links

Robert Vandermolen's $\overrightarrow{\mathbf{N}}$ and $\overrightarrow{\mathbf{B}}$ slides:
`https://docs.google.com/presentation/d/12srwxuU-ecRSIezuA9kGSNdzopTR5IVTdxUKG2Gs7Ec/edit#slide=id.g1de4eec692_1_20`

11   §14.1 Functions of Several Variables

Up until now, our functions have all been defined parametrically with an extra variable t as the independent function. For much of the rest of the course, we will now be considering functions of several variables, i.e., things like

$$z = f(x, y) \qquad \text{or} \qquad w = f(x, y, z).$$

In the case that z is described as a function of x and y, i.e., $z = f(x, y)$, we say that x and y are *independent variables* and z is the *dependent* variable, because the value of z depends on what the values of x and y are.

Domains

In the single variable case (Calculus 1 and 2), the domain of a function was a portion of the real line. For example,

$$f(x) = \frac{1}{x - 1}$$

has domain $(-\infty, 1) \cup (1, \infty)$, or all of $\mathbb{R}$ except the number 1. This is contrasted with the multivariable case, where the domain of $f(x, y)$ is a portion of the xy-plane.

Example. *Find the domain of $f(x, y) = \sqrt{y - x^2}$.*

To find the domain, we are asking the question, "What points (x, y) can I plug into this function?" For example, we cannot plug in points that give us a negative number under a square root. So something like $(3, 4)$ isn't in the domain of $f(x, y)$ because $f(3, 4) = \sqrt{4 - 3^2} = \sqrt{-5}$, which isn't defined. On the other hand, $(0, 3)$ is in the domain of $f(x, y)$ because $f(0, 3) = \sqrt{3}$ which is a perfectly valid real number.

The defining restriction for $f(x, y)$ in this case is that we must have a nonnegative number under the square root. As an inequality, this means
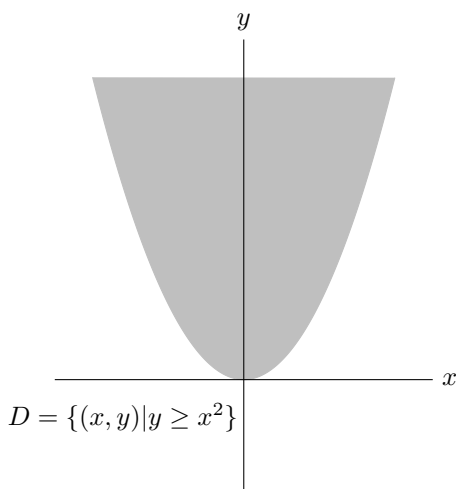
$$y - x^2 \geq 0.$$

We include 0 (the '=' part of the inequality) because $\sqrt{0} = 0$ is a real number.

Let's visualize the defining inequality. We can isolate y and then graph the resulting inequality:

$$y - x^2 \geq 0 \Rightarrow y \geq x^2.$$

Thus the domain is all the points (x, y) on or above the graph of $y = x^2$.

$$D = \{(x, y) | y \geq x^2\}$$

Any points in the gray region can be plugged into $f(x, y)$ and any points outside the gray region cannot be plugged into $f(x, y)$.

This example had a square root in it. Another common operation with a restricted domain is the inversion function $1/x$, i.e., a function with something in its denominator. The next example demonstrates this.

Example. *Find the domain of the function $z = \frac{1}{xy}$.*

In this case the defining restriction is that we must not have 0 in the denominator. As an inequality, this means

$$xy \neq 0 \Rightarrow x \neq 0 \text{ or } y \neq 0.$$

The lines $x = 0$ and $y = 0$ are the y- and the x-axis respectively. So the inequalities $x \neq 0$ or $y \neq 0$ are all the points not on the x- or y-axes.
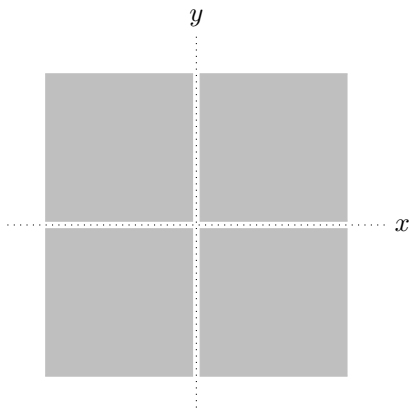
$$D = \{(x, y) | xy \neq 0\}$$

Ranges

The range of a function $z = f(x, y)$ is calculated in the same way as a single-variable functions. We ask the question, "What z-values (i.e., heights) can be obtained from this function?" These are still subsets of the number line, so they look more familiar than domains do.

Example. *Find the range of* $f(x, y) = \sqrt{y - x^2}$ *and* $z = \frac{1}{xy}$.

In the first case, we know that square roots are always nonnegative, so we can exclude any negative numbers from the range. We can also include 0 and every positive number in the range, because we can obtain arbitrarily large values of $y - x^2$ (the value under the square root) and hence arbitrarily large values of $f(x, y)$. So the range of $f(x, y)$ is $[0, \infty)$.

In the second case, we are considering a fraction $\frac{1}{xy}$. We can have any nonzero value for xy, so we can get any z-value close to 0 with a large value for xy. We can also get any large z-value with a value for xy between 0 and 1. (This argument works on the negative side as well.) Thus we can get any number except 0. So the range of z is $(-\infty, 0) \cup (0, \infty)$.

Level Curves

To help ourselves graph multivariable functions, we will often consider the level curves of a function.

Definition. *Let c be a real number. The set of points (x, y) where $f(x, y) = c$ is called a <u>level curve</u> of f.*

A level curve can be thought of as the points where a function $f(x, y)$ obtains the height $z = c$.

Example. *Let $f(x, y) = 100 - x^2 - y^2$. Find the level curves for $z = 100, z = 75, z = 51, and z = 0$.*
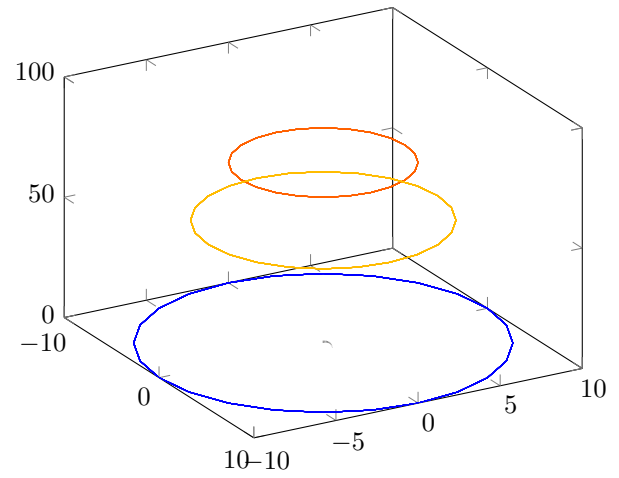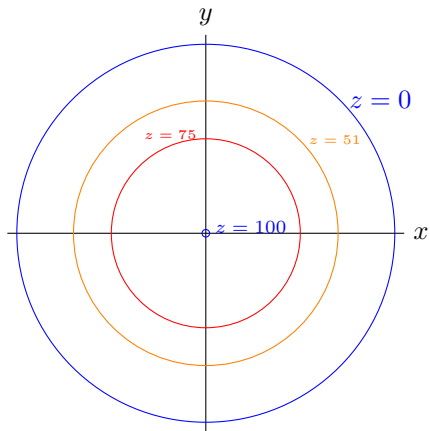
For $z = 100$, we get $100 = 100 - x^2 - y^2$, or $x^2 + y^2 = 0$. This defines the point $(0, 0, 0)$.
For $z = 75$, we get $75 = 100 - x^2 - y^2$, or $25 = x^2 + y^2$. This defines the circle of radius 5 centered at $(0, 0, 75)$, lying in the plane $z = 75$.
For $z = 51$, we get $51 = 100 - x^2 - y^2$, or $49 = x^2 + y^2$. This defines the circle of radius 7 centered at $(0, 0, 51)$, lying in the plane $z = 51$.
For $z = 0$, we get $0 = 100 - x^2 - y^2$, or $100 = x^2 + y^2$. This defines the circle of radius 10 centered at $(0, 0, 0)$, lying in the plane $z = 0$.
We can visualize these in only the xy-plane, or in space at the appropriate heights.

12   §14.2 Limits and Continuity